Triangle element for plane strain hyperelastic FEA

Vinicius F. Arcaro, UNICAMP/FEC

1 Geometry

Figure 1 shows the geometry for a triangle element.

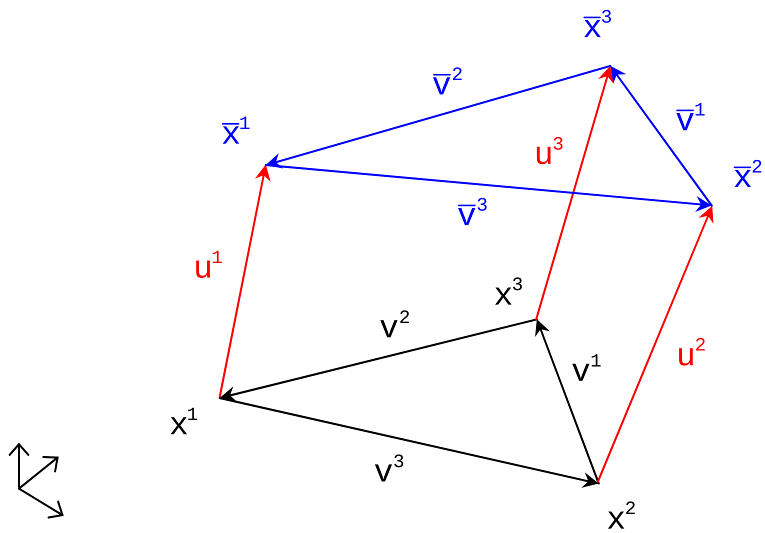

Figure 1

$$\bar{v}^1 = v^1 + u^3 - u^2$$

$$\bar{v}^2 = v^2 + u^1 - u^3$$

$$\bar{v}^3 = v^3 + u^2 - u^1$$

$$w = \frac{v^1 \times v^2}{\left\| v^1 \times v^2 \right\|}$$

$$\alpha = w^T \left(v^1 \times v^2 \right)$$

$$\bar{w} = \frac{\bar{v}^1 \times \bar{v}^2}{\left\| \bar{v}^1 \times \bar{v}^2 \right\|}$$

$$\bar{\alpha} = \bar{w}^T \left(\bar{v}^1 \times \bar{v}^2 \right)$$

$$w^i = w \times v^i \quad , \quad i = 1, 2, 3$$

The unit vectors w and $\bar{w}$ are orthogonal to the element’s surface in the undeformed state and deformed state respectively. Notice that these vectors points toward the observer, when nodes associated with the element appear counterclockwise. It is important to emphasize that $\bar{w} = w$ for the plane strain case.

The scalars α and $\bar{\alpha}$ are equal to twice the area of the element in the undeformed state and deformed state respectively.

The scalars δ and $\bar{\delta}$ are equal to the thickness of the element in the undeformed state and deformed state respectively.

2 Deformation gradient tensor

The deformation gradient tensor can be written as:

$$F = I + \frac{1}{\alpha}\left[u^1\left(w^1\right)^T + u^2\left(w^2\right)^T + u^3\left(w^3\right)^T\right]$$

Notice that,

$$Fv^1 = \bar{v}^1$$

$$Fv^2 = \bar{v}^2$$

$$Fv^3 = \bar{v}^3$$

$$Fw = w$$

The invariants of the deformation gradient tensor and its derivatives with respect to the nodal displacements of the element can be written as:

2.1 Invariant 1

$$f_1 = tr\left(F\right) = 1 + \frac{1}{\alpha}\left[\left(w^1\right)^T \bar{v}^2 - \left(w^2\right)^T \bar{v}^1\right]$$

$$\frac{\partial f_1}{\partial u^1} = \frac{1}{\alpha} w^1$$

$$\frac{\partial f_1}{\partial u^2} = \frac{1}{\alpha} w^2$$

2.2 Invariant 2

$$f_2 = \mathrm{tr}\left(F^T F\right) = 1 + \frac{1}{\alpha^2}\left[\left(\bar{v}^2\right)^T \bar{z}^1 - \left(\bar{v}^1\right)^T \bar{z}^2\right]$$

$$\frac{\partial f_2}{\partial u^1} = \frac{2}{\alpha^2}\bar{z}^1$$

$$\frac{\partial f_2}{\partial u^2} = \frac{2}{\alpha^2}\bar{z}^2$$

Where,

$$\bar{z}^1 = \left[\left(v^1\right)^T v^1 \bar{v}^2 - \left(v^1\right)^T v^2 \bar{v}^1\right]$$

$$\bar{z}^2 = \left[\left(v^2\right)^T v^1 \bar{v}^2 - \left(v^2\right)^T v^2 \bar{v}^1\right]$$

2.3 Invariant 3

$$f_3 = \det\left(F\right) = \frac{\bar{\alpha}}{\alpha}$$

$$\frac{\partial f_3}{\partial u^1} = \frac{1}{\alpha}\left(w \times \bar{v}^1\right)$$

$$\frac{\partial f_3}{\partial u^2} = \frac{1}{\alpha}\left(w \times \bar{v}^2\right)$$

3 Right Cauchy-Green deformation tensor

The right Cauchy-Green deformation tensor can be written in terms of the deformation gradient tensor as:

$$C = F^T F$$

The invariants of the right Cauchy-Green deformation tensor and its derivatives with respect to the nodal displacements of the element can be written as:

3.1 Invariant 1

$$c_1 = \mathrm{tr}\,(C) = 1 + \frac{1}{\alpha^2}\left[\left(\overline{v}^2\right)^T \overline{z}^1 - \left(\overline{v}^1\right)^T \overline{z}^2\right]$$

$$\frac{\partial c_1}{\partial u^1} = \frac{2}{\alpha^2}\,\overline{z}^1$$

$$\frac{\partial c_1}{\partial u^2} = \frac{2}{\alpha^2}\,\overline{z}^2$$

3.2 Invariant 2

$$\begin{aligned}
c_2 &= \mathrm{tr}\left(C^T C\right) = \\
&+1 + \frac{2}{\alpha^2}\left(w^T\overline{z}^1 w^T\overline{v}^2 - w^T\overline{z}^2 w^T\overline{v}^1\right) + \\
&+ \frac{1}{\alpha^4}\left[\left(\overline{v}^1\right)^T \overline{v}^1 \left(\overline{z}^2\right)^T \overline{z}^2 - \left(\overline{v}^1\right)^T \overline{v}^2 \left(\overline{z}^2\right)^T \overline{z}^1\right] + \\
&+ \frac{1}{\alpha^4}\left[\left(\overline{v}^2\right)^T \overline{v}^2 \left(\overline{z}^1\right)^T \overline{z}^1 - \left(\overline{v}^1\right)^T \overline{v}^2 \left(\overline{z}^2\right)^T \overline{z}^1\right]
\end{aligned}$$

$$\begin{aligned}
\frac{\partial c_2}{\partial u^1} &= \\
&+ \frac{4}{\alpha^4}\left[\left(\overline{z}^1\right)^T \overline{z}^1\overline{v}^2 - \left(\overline{z}^2\right)^T \overline{z}^1\overline{v}^1\right] + \frac{4}{\alpha^2}\,w^T\overline{z}^1 w = \\
&+ \frac{4}{\alpha^4}\left[\left(\overline{v}^2\right)^T \overline{z}^1\overline{z}^1 - \left(\overline{v}^1\right)^T \overline{z}^1\overline{z}^2\right] + \frac{4}{\alpha^2}\,w^T\overline{z}^1 w
\end{aligned}$$

$$\begin{aligned}
\frac{\partial c_2}{\partial u^2} &= \\
&+ \frac{4}{\alpha^4}\left[\left(\overline{z}^2\right)^T \overline{z}^1\overline{v}^2 - \left(\overline{z}^2\right)^T \overline{z}^2\overline{v}^1\right] + \frac{4}{\alpha^2}\,w^T\overline{z}^2 w = \\
&+ \frac{4}{\alpha^4}\left[\left(\overline{v}^2\right)^T \overline{z}^2\overline{z}^1 - \left(\overline{v}^1\right)^T \overline{z}^2\overline{z}^2\right] + \frac{4}{\alpha^2}\,w^T\overline{z}^2 w
\end{aligned}$$

3.3 Invariant 3

$$c_3 = \det(C) = \left(\frac{\bar{\alpha}}{\alpha}\right)^2$$

$$\frac{\partial c_3}{\partial u^1} = \frac{2\bar{\alpha}}{\alpha^2}\left(w \times \bar{\nabla}^1\right)$$

$$\frac{\partial c_3}{\partial u^2} = \frac{2\bar{\alpha}}{\alpha^2}\left(w \times \bar{\nabla}^2\right)$$

4 Left Cauchy-Green deformation tensor

The left Cauchy-Green deformation tensor can be written in terms of the deformation gradient tensor as:

$$B = FF^T$$

It is easy to show that the invariants of the left Cauchy-Green deformation tensor are identical to the invariants of the right Cauchy-Green deformation tensor. The traces of the first four powers of the left Cauchy-Green deformation tensor are required to evaluate the coefficients of the characteristic equation of the Cauchy stress tensor.

$$\det(B) = \det\left(FF^T\right) = f_3^2$$

$$\mathrm{tr}(B) = \mathrm{tr}\left(FF^T\right) = \mathrm{tr}\left(F^TF\right) = c_1$$

$$\mathrm{tr}(BB) = \mathrm{tr}\left(FF^TFF^T\right) = \mathrm{tr}\left(F^TFF^TF\right) = c_2$$

The Cayley-Hamilton Theorem states that every matrix satisfies its own characteristic equation.

$$-BBB + \mathrm{tr}(B)\,BB - \frac{1}{2}\left[\mathrm{tr}^2(B) - \mathrm{tr}(BB)\right]B + \det(B)\,I = 0$$

Therefore,

$$BBB = \mathrm{tr}(B)\,BB - \frac{1}{2}\left[\mathrm{tr}^2(B) - \mathrm{tr}(BB)\right]B + \det(B)\,I$$

$$BBBB = \mathrm{tr}(B)\,BBB - \frac{1}{2}\left[\mathrm{tr}^2(B) - \mathrm{tr}(BB)\right]BB + \det(B)\,B$$

$$\mathrm{tr}(BBB) = \frac{3}{2}c_1c_2 - \frac{1}{2}c_1^3 + 3f_3^2$$

$$\mathrm{tr}(BBBB) = c_1^2c_2 + \frac{1}{2}c_2^2 - \frac{1}{2}c_1^4 + 4f_3^2c_1$$

5 Strain energy

Consider ψ as the strain energy function per unit undeformed volume.

$$\psi = \psi(c_1, c_2, c_3)$$

$$\psi_i = \frac{\partial\psi}{\partial c_i} \quad , \; i = 1, 2, 3$$

6 Total potential energy

Consider ω as the work done by external forces. The total potential energy ϕ can be written as a function of the unknown displacements by a summation over all elements.

$$\phi = \sum \psi \frac{\alpha\delta}{2} - \omega$$

The gradient of the total potential energy can be written as a function of the unknown displacements by a summation over all elements.

$$\nabla\phi = \sum (\nabla\psi) \frac{\alpha\delta}{2} - \nabla\omega$$

The gradient of the strain energy function for the element is calculated using the chain rule and the gradients of the invariants of the right Cauchy-Green deformation tensor.

$$\frac{\partial\psi}{\partial u^i}\frac{\alpha\delta}{2} = \left(\psi_1 \frac{\partial c_1}{\partial u^i} + \psi_2 \frac{\partial c_2}{\partial u^i} + \psi_3 \frac{\partial c_3}{\partial u^i}\right)\frac{\alpha\delta}{2} \quad , \; i = 1, 2, 3$$

6.1 Derivatives with respect to displacements

$$\phi = \phi\left(u^1, u^2, u^3\right)$$

$$\left.\begin{aligned} \bar{v}^1 &= v^1 + u^3 - u^2 \\ \bar{v}^2 &= v^2 + u^1 - u^3 \end{aligned}\right\} \Rightarrow$$

$$\frac{\partial\phi}{\partial u^1} = +\frac{\partial\phi}{\partial\bar{v}^2}$$

$$\frac{\partial\phi}{\partial u^2} = -\frac{\partial\phi}{\partial\bar{v}^1}$$

$$\frac{\partial\phi}{\partial u^3} = +\frac{\partial\phi}{\partial\bar{v}^1} - \frac{\partial\phi}{\partial\bar{v}^2} = -\left(\frac{\partial\phi}{\partial u^1} + \frac{\partial\phi}{\partial u^2}\right)$$

7 Cauchy stress tensor

The Cauchy stress tensor can be written as:

$$S = \frac{2}{f_3} F \frac{\partial\psi}{\partial C} F^T$$

However,

$$\frac{\partial\psi}{\partial C} = \psi_1 \frac{\partial c_1}{\partial C} + \psi_2 \frac{\partial c_2}{\partial C} + \psi_3 \frac{\partial c_3}{\partial C}$$

$$c_1 = \mathrm{tr}(C) \Rightarrow \frac{\partial c_1}{\partial C} = I$$

$$c_2 = \mathrm{tr}\left(C^T C\right) \Rightarrow \frac{\partial c_2}{\partial C} = 2C$$

$$c_3 = \det(C) \Rightarrow \frac{\partial c_3}{\partial C} = f_3^2 C^{-1}$$

Therefore,

$$S = \frac{2\psi_1}{f_3} B + \frac{4\psi_2}{f_3} BB + 2\psi_3 f_3 I$$

8 Stress orthogonal to the element’s surface

The traction vector related to the unit vector orthogonal to the element’s surface in the deformed state can be written as:

$$S\bar{w} = \frac{2\psi_1}{f_3} B\bar{w} + \frac{4\psi_2}{f_3} BB\bar{w} + 2\psi_3 f_3 \bar{w}$$

$$F^T\bar{w} = w^T\bar{w}w \Rightarrow B\bar{w} = w^T\bar{w}w$$

However, for the plane strain case the vectors w and $\bar{w}$ are equal. Therefore,

$$\bar{w} = w \Rightarrow$$

$$Sw = \left(\frac{2\psi_1}{f_3} + \frac{4\psi_2}{f_3} + 2\psi_3 f_3\right) w$$

The unit vector w is a principal direction associated with a principal stress given by:

$$\sigma_3 = \frac{2\psi_1}{f_3} + \frac{4\psi_2}{f_3} + 2\psi_3 f_3$$

9 Stress parallel to the element’s surface

The characteristic equation for the Cauchy stress tensor can be written as:

$$\det(S - \sigma I) = -\sigma^3 + \operatorname{tr}(S)\,\sigma^2 - \frac{1}{2}\left[\operatorname{tr}^2(S) - \operatorname{tr}(SS)\right]\sigma + \det(S) = 0$$

The principal stress orthogonal to the element’s surface in the deformed state is equal to σ_3. Therefore, the characteristic equation reduces to:

$$\sigma^2 - \left[\operatorname{tr}(S) - \sigma_3\right]\sigma + \frac{1}{2}\left[\operatorname{tr}^2(S) - \operatorname{tr}(SS)\right] - \left[\operatorname{tr}(S) - \sigma_3\right]\sigma_3 = 0$$

The Cauchy stress tensor can be written in terms of the left Cauchy-Green deformation tensor as:

$$S = \frac{2\psi_1}{f_3} B + \frac{4\psi_2}{f_3} BB + 2\psi_3 f_3 I$$

The traces of the first two powers of the Cauchy stress tensor, required to evaluate the coefficients of its characteristic equation, can be written as:

$$tr(S) = \frac{2\psi_1}{f_3} c_1 + \frac{4\psi_2}{f_3} c_2 + 6\psi_3 f_3$$

$$tr(SS) =$$

$$+4\frac{\psi_1\psi_1}{f_3^2} c_2 +$$

$$+8\frac{\psi_1\psi_2}{f_3^2}\left(3c_1c_2 - c_1^3 + 6f_3^2\right) +$$

$$+8\psi_1\psi_3 c_1 +$$

$$+8\frac{\psi_2\psi_2}{f_3^2}\left(2c_1^2c_2 + c_2^2 - c_1^4 + 8c_1f_3^2\right) +$$

$$+16\psi_2\psi_3 c_2 +$$

$$+12\psi_3\psi_3 f_3^2$$

10 Strain energy

The derivatives of the strain energy with respect to the nodal displacements of the element can be written as:

$$\frac{\partial\psi}{\partial u^1}\frac{\alpha\delta}{2} =$$

$$+\frac{\psi_1}{\alpha}\bar{z}^1\delta +$$

$$+2\psi_2\left\{\frac{1}{\alpha^3}\left[\left(\bar{z}^1\right)^T \bar{z}^1\bar{v}^2 - \left(\bar{z}^2\right)^T \bar{z}^1\bar{v}^1\right] + \frac{1}{\alpha} w^T\bar{z}^1 w\right\}\delta +$$

$$+\psi_3\left(\frac{\bar{\alpha}}{\alpha}\right)\left(w \times \bar{v}^1\right)\delta$$

$$\frac{\partial\psi}{\partial u^2}\frac{\alpha\delta}{2} =$$

$$+\frac{\psi_1}{\alpha}\bar{z}^2\delta +$$

$$+2\psi_2\left\{\frac{1}{\alpha^3}\left[\left(\bar{z}^2\right)^T \bar{z}^1\bar{v}^2 - \left(\bar{z}^2\right)^T \bar{z}^2\bar{v}^1\right] + \frac{1}{\alpha} w^T\bar{z}^2 w\right\}\delta +$$

$$+\psi_3\left(\frac{\bar{\alpha}}{\alpha}\right)\left(w \times \bar{v}^2\right)\delta$$

11 Examples

Example 1: Consider a 10 x 10 square surface made of compressible Neo-Hookean material. The opposite vertical sides of the square surface are pulled apart with relative displacement equal to 3. The strain energy function for this material is given by:

$$\psi = \frac{\mu}{2} c_1 \left(c_3\right)^{-\frac{1}{3}} + \frac{\kappa}{2}\left[c_3 - 2\left(c_3\right)^{\frac{1}{2}}\right] + \frac{1}{2}\left(\kappa - 3\mu\right)$$

$\mu = 0.4225$, $\kappa = 5.0000$

Figure 2 shows the meshes for the initial and final surfaces.

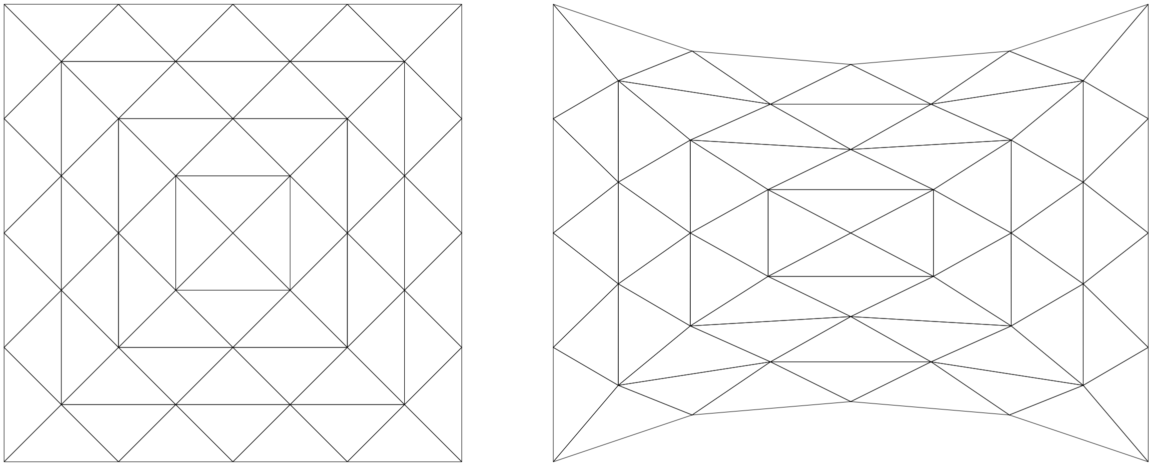

Figure 2

Figure 3 shows the deformed shape by ANSYS.

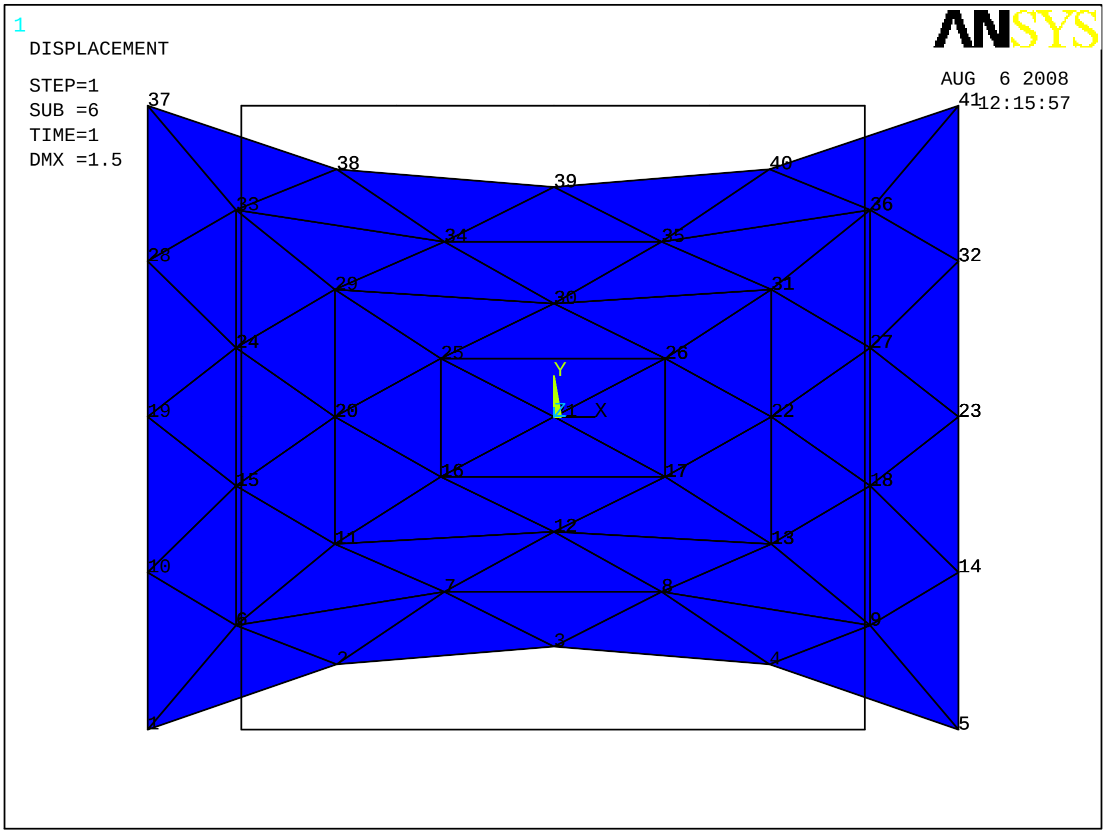

Figure 3

Table 1 shows some displacement results compared with ANSYS.

Table 1

Node	Displ Y	ANSYS	Error (%)
2	1.0236	1.0236	0.00
3	1.3156	1.3156	0.00
4	1.0236	1.0236	0.00

Table 2 shows the results for principal stress parallel to the element's surface compared with ANSYS.

Table 2

	Stress	ANSYS	Error (%)
Max	0.973247	0.973247	0.00
Min	-0.511450	-0.511450	0.00

Example 2: Consider a 10 x 10 square surface made of compressible Mooney-Rivlin material. The opposite vertical

sides of the square surface are pulled apart with relative displacement equal to 3. The strain energy function for this material is given by:

$$\psi = \left[\mu_{10}c_1 + \frac{1}{2}\mu_{01}\left(c_1^2 - c_2\right)\left(c_3\right)^{-\frac{1}{3}}\right]\left(c_3\right)^{-\frac{1}{3}} + \frac{\kappa}{2}\left[c_3 - 2\left(c_3\right)^{\frac{1}{2}}\right] +$$

$$+\frac{\kappa}{2} - 3\left(\mu_{10} + \mu_{01}\right)$$

$$\mu_{10} = 0.3750 \ , \ \mu_{01} = -0.1250 \ , \ \kappa = 5.0000$$

Figure 4 shows the meshes for the initial and final surfaces.

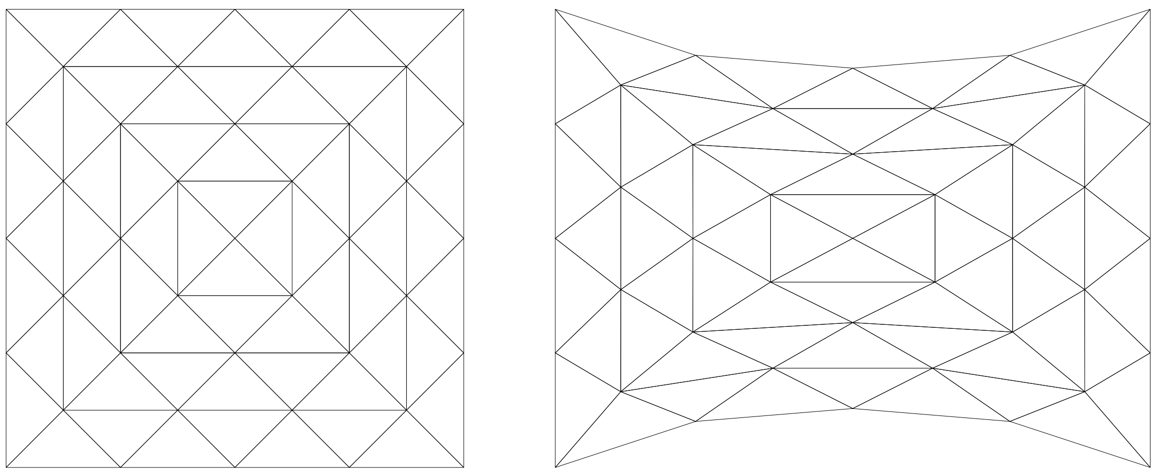

Figure 4

Figure 5 shows the deformed shape by ANSYS.

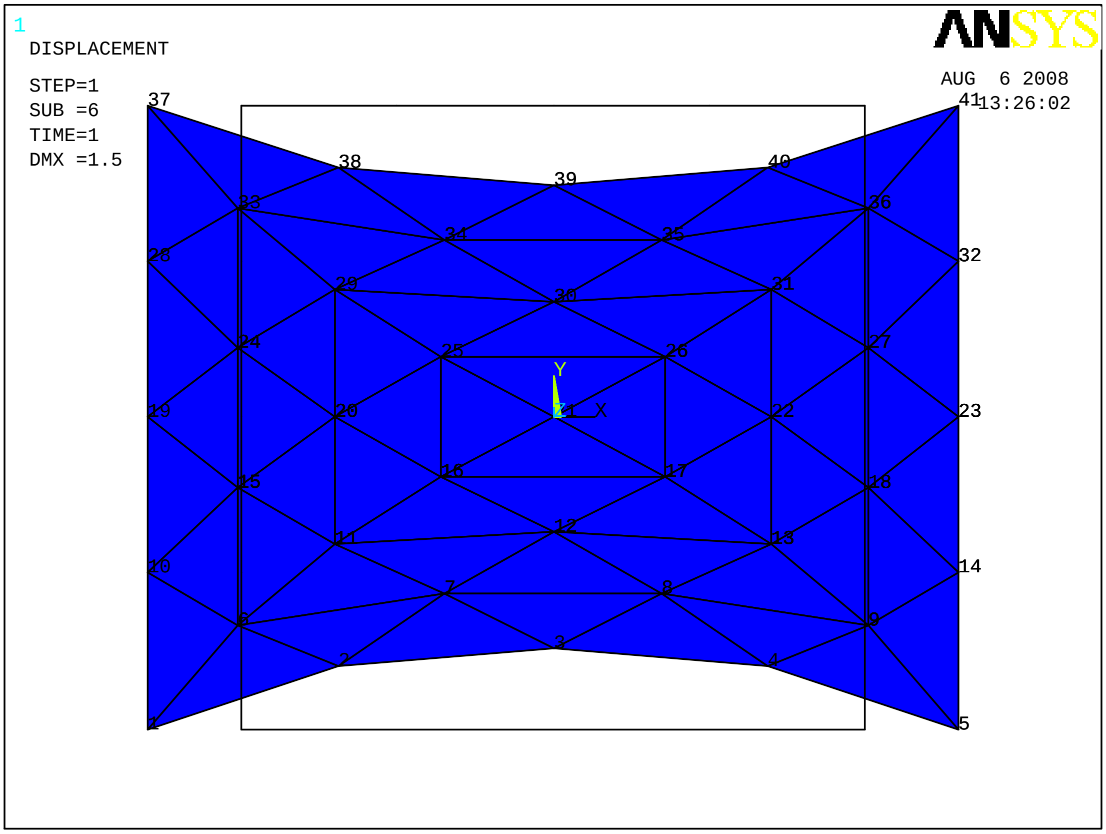

Figure 5

Table 3 shows some displacement results compared with ANSYS.

Table 3

Node	Displ Y	ANSYS	Error (%)
2	1.0048	1.0048	0.00
3	1.2861	1.2861	0.00
4	1.0048	1.0048	0.00

Table 4 shows the results for principal stress parallel to the element’s surface compared with ANSYS.

Table 4

	Stress	ANSYS	Error (%)
Max	1.055990	1.056000	0.00
Min	-0.490209	-0.490208	0.00