Triangle element for plane stress hyperelastic FEA

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1 Geometry

Figure 1 shows the geometry for a triangle element.

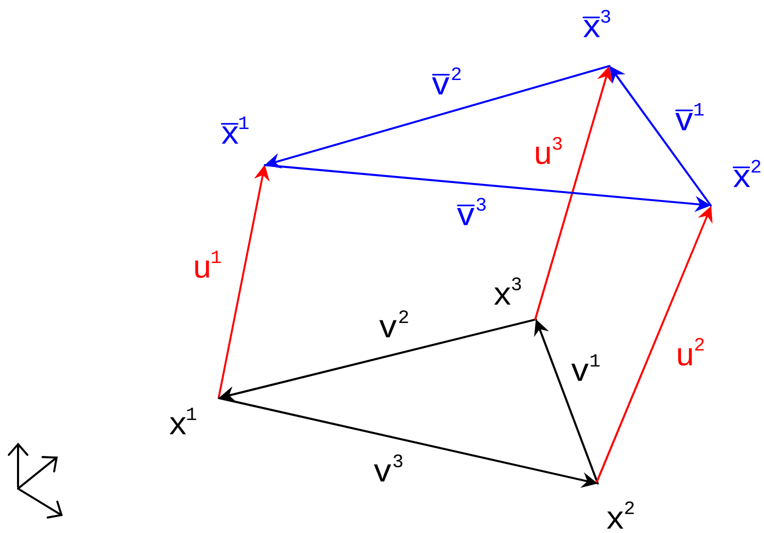

Figure 1

$$\bar{v}^1 = v^1 + u^3 - u^2$$

$$\bar{v}^2 = v^2 + u^1 - u^3$$

$$\bar{v}^3 = v^3 + u^2 - u^1$$

$$w = \frac{v^1 \times v^2}{\left\| v^1 \times v^2 \right\|}$$

$$\alpha = w^T \left(v^1 \times v^2 \right)$$

$$\bar{w} = \frac{\bar{v}^1 \times \bar{v}^2}{\left\| \bar{v}^1 \times \bar{v}^2 \right\|}$$

$$\bar{\alpha} = \bar{w}^T \left(\bar{v}^1 \times \bar{v}^2 \right)$$

$$w^i = w \times v^i \quad , \; i = 1, 2, 3$$

The unit vectors w and $\bar{w}$ are orthogonal to the element's surface in the undeformed state and deformed state respectively. Notice that these vectors points toward the observer, when nodes associated with the element appear counterclockwise.

The scalars α and $\bar{\alpha}$ are equal to twice the area of the element in the undeformed state and deformed state respectively.

The scalars δ and $\bar{\delta}$ are equal to the thickness of the element in the undeformed state and deformed state respectively.

2 Deformation gradient tensor

The deformation gradient tensor can be written as:

$$\hat{F} = I + \frac{1}{\alpha}\left[u^1\left(w^1\right)^T + u^2\left(w^2\right)^T + u^3\left(w^3\right)^T\right] + \left(\frac{\bar{\delta}}{\delta}\bar{w} - w\right)w^T$$

Notice that,

$$\hat{F}v^1 = \bar{v}^1$$

$$\hat{F}v^2 = \bar{v}^2$$

$$\hat{F}v^3 = \bar{v}^3$$

$$\hat{F}\left(\delta w\right) = \left(\bar{\delta}\bar{w}\right)$$

The invariants of the deformation gradient tensor can be written as:

2.1 Invariant 1

$$\hat{f}_1 = tr\left(\hat{F}\right) = \frac{1}{\alpha}\left[\left(w^1\right)^T\bar{v}^2 - \left(w^2\right)^T\bar{v}^1\right] + \frac{\bar{\delta}}{\delta}w^T\bar{w}$$

2.2 Invariant 2

$$\hat{f}_2 = tr\left(\hat{F}^T\hat{F}\right) = \frac{1}{\alpha^2}\left[\left(\bar{v}^2\right)^T\bar{z}^1 - \left(\bar{v}^1\right)^T\bar{z}^2\right] + \left(\frac{\bar{\delta}}{\delta}\right)^2$$

Where,

$$\bar{z}^1 = \left[\left(v^1\right)^T v^1 \bar{v}^2 - \left(v^1\right)^T v^2 \bar{v}^1 \right]$$

$$\bar{z}^2 = \left[\left(v^2\right)^T v^1 \bar{v}^2 - \left(v^2\right)^T v^2 \bar{v}^1 \right]$$

2.3 Invariant 3

$$\hat{f}_3 = \det\left(\hat{F}\right) = \frac{\bar{\delta}\bar{\alpha}}{\delta\alpha}$$

3 Right Cauchy-Green deformation tensor

The right Cauchy-Green deformation tensor can be written in terms of the deformation gradient tensor as:

$$\hat{C} = \hat{F}^T \hat{F}$$

The invariants of the right Cauchy-Green deformation tensor can be written as:

$$\hat{c}_1 = \mathrm{tr}\left(\hat{C}\right) = \frac{1}{\alpha^2} \left[\left(\bar{v}^2\right)^T \bar{z}^1 - \left(\bar{v}^1\right)^T \bar{z}^2 \right] + \left(\frac{\bar{\delta}}{\delta}\right)^2$$

$$\hat{c}_2 = \mathrm{tr}\left(\hat{C}^T \hat{C}\right) =$$

$$+ \frac{1}{\alpha^4} \left[\left(\bar{v}^1\right)^T \bar{v}^1 \left(\bar{z}^2\right)^T \bar{z}^2 - \left(\bar{v}^1\right)^T \bar{v}^2 \left(\bar{z}^2\right)^T \bar{z}^1 \right] +$$

$$+ \frac{1}{\alpha^4} \left[\left(\bar{v}^2\right)^T \bar{v}^2 \left(\bar{z}^1\right)^T \bar{z}^1 - \left(\bar{v}^1\right)^T \bar{v}^2 \left(\bar{z}^2\right)^T \bar{z}^1 \right] +$$

$$+ \left(\frac{\bar{\delta}}{\delta}\right)^4$$

$$\hat{c}_3 = \det\left(\hat{C}\right) = \left(\frac{\bar{\delta}\bar{\alpha}}{\delta\alpha}\right)^2$$

4 Left Cauchy-Green deformation tensor

The left Cauchy-Green deformation tensor can be written in terms of the deformation gradient tensor as:

$$\hat{B} = \hat{F}\hat{F}^{T}$$

It is easy to show that the invariants of the left Cauchy-Green deformation tensor are identical to the invariants of the right Cauchy-Green deformation tensor. The traces of the first four powers of the left Cauchy-Green deformation tensor are required to evaluate the coefficients of the characteristic equation of the Cauchy stress tensor.

$$\det\left(\hat{B}\right) = \det\left(\hat{F}\hat{F}^{T}\right) = \hat{f}_3^2$$

$$\mathrm{tr}\left(\hat{B}\right) = \mathrm{tr}\left(\hat{F}\hat{F}^{T}\right) = \mathrm{tr}\left(\hat{F}^{T}\hat{F}\right) = \hat{c}_1$$

$$\mathrm{tr}\left(\hat{B}\hat{B}\right) = \mathrm{tr}\left(\hat{F}\hat{F}^{T}\hat{F}\hat{F}^{T}\right) = \mathrm{tr}\left(\hat{F}^{T}\hat{F}\hat{F}^{T}\hat{F}\right) = \hat{c}_2$$

The Cayley-Hamilton Theorem states that every matrix satisfies its own characteristic equation.

$$-\hat{B}\hat{B}\hat{B} + \mathrm{tr}\left(\hat{B}\right)\hat{B}\hat{B} - \frac{1}{2}\left[\mathrm{tr}^2\left(\hat{B}\right) - \mathrm{tr}\left(\hat{B}\hat{B}\right)\right]\hat{B} + \det\left(\hat{B}\right)I = 0$$

Therefore,

$$\hat{B}\hat{B}\hat{B} = \mathrm{tr}\left(\hat{B}\right)\hat{B}\hat{B} - \frac{1}{2}\left[\mathrm{tr}^2\left(\hat{B}\right) - \mathrm{tr}\left(\hat{B}\hat{B}\right)\right]\hat{B} + \det\left(\hat{B}\right)I$$

$$\hat{B}\hat{B}\hat{B}\hat{B} = \mathrm{tr}\left(\hat{B}\right)\hat{B}\hat{B}\hat{B} - \frac{1}{2}\left[\mathrm{tr}^2\left(\hat{B}\right) - \mathrm{tr}\left(\hat{B}\hat{B}\right)\right]\hat{B}\hat{B} + \det\left(\hat{B}\right)\hat{B}$$

$$\mathrm{tr}\left(\hat{B}\hat{B}\hat{B}\right) = -\frac{1}{2}\hat{c}_1^3 + \frac{3}{2}\hat{c}_1\hat{c}_2 + 3\hat{f}_3^2$$

$$\mathrm{tr}\left(\hat{B}\hat{B}\hat{B}\hat{B}\right) = -\frac{1}{2}\hat{c}_1^4 + \hat{c}_1^2\hat{c}_2 + \frac{1}{2}\hat{c}_2^2 + 4\hat{c}_1\hat{f}_3^2$$

5 Strain energy

Consider ψ as the strain energy function per unit undeformed volume.

$$\psi = \psi\left(\hat{c}_1, \hat{c}_2, \hat{c}_3\right)$$

$$\psi_i = \frac{\partial\psi}{\partial\hat{c}_i} \quad , \; i = 1, 2, 3$$

6 Total potential energy

Consider ω as the work done by external forces. The total potential energy ϕ can be written as a function of the unknown displacements by a summation over all elements.

$$\phi = \sum \psi \frac{\alpha\delta}{2} - \omega$$

The gradient of the total potential energy can be written as a function of the unknown displacements by a summation over all elements.

$$\nabla\phi = \sum (\nabla\psi) \frac{\alpha\delta}{2} - \nabla\omega$$

The gradient of the strain energy function for the element is calculated using the chain rule and the gradients of the invariants of the right Cauchy-Green deformation tensor.

$$\frac{\partial\psi}{\partial u^i}\frac{\alpha\delta}{2} = \left(\psi_1 \frac{\partial\hat{c}_1}{\partial u^i} + \psi_2 \frac{\partial\hat{c}_2}{\partial u^i} + \psi_3 \frac{\partial\hat{c}_3}{\partial u^i}\right)\frac{\alpha\delta}{2} \quad , \; i = 1, 2, 3$$

6.1 Derivatives with respect to displacements

$$\phi = \phi\left(u^1, u^2, u^3\right)$$

$$\left.\begin{aligned} \bar{v}^1 &= v^1 + u^3 - u^2 \\ \bar{v}^2 &= v^2 + u^1 - u^3 \end{aligned}\right\} \Rightarrow$$

$$\frac{\partial\phi}{\partial u^1} = + \frac{\partial\phi}{\partial\bar{v}^2}$$

$$\frac{\partial \phi}{\partial u^2} = -\frac{\partial \phi}{\partial \nabla^1}$$

$$\frac{\partial \phi}{\partial u^3} = +\frac{\partial \phi}{\partial \nabla^1} - \frac{\partial \phi}{\partial \nabla^2} = -\left(\frac{\partial \phi}{\partial u^1} + \frac{\partial \phi}{\partial u^2}\right)$$

7 Cauchy stress tensor

The Cauchy stress tensor can be written as:

$$S = \frac{2}{\hat{f}_3}\hat{F}\frac{\partial \psi}{\partial \hat{C}}\hat{F}^T$$

However,

$$\frac{\partial \psi}{\partial \hat{C}} = \psi_1 \frac{\partial \hat{c}_1}{\partial \hat{C}} + \psi_2 \frac{\partial \hat{c}_2}{\partial \hat{C}} + \psi_3 \frac{\partial \hat{c}_3}{\partial \hat{C}}$$

$$\hat{c}_1 = \mathrm{tr}\left(\hat{C}\right) \Rightarrow \frac{\partial \hat{c}_1}{\partial \hat{C}} = I$$

$$\hat{c}_2 = \mathrm{tr}\left(\hat{C}^T\hat{C}\right) \Rightarrow \frac{\partial \hat{c}_2}{\partial \hat{C}} = 2\hat{C}$$

$$\hat{c}_3 = \det\left(\hat{C}\right) \Rightarrow \frac{\partial \hat{c}_3}{\partial \hat{C}} = \hat{f}_3^2\hat{C}^{-1}$$

Therefore,

$$S = \frac{2\psi_1}{\hat{f}_3}\hat{B} + \frac{4\psi_2}{\hat{f}_3}\hat{B}\hat{B} + 2\psi_3\hat{f}_3 I$$

8 Stress parallel to the element’s surface

The characteristic equation for the Cauchy stress tensor can be written as:

$$\det(S - \sigma I) = -\sigma^3 + \mathrm{tr}(S)\,\sigma^2 - \frac{1}{2}\left[\mathrm{tr}^2(S) - \mathrm{tr}(SS)\right]\sigma + \det(S) = 0$$

The principal stress orthogonal to the element's surface in the deformed state is equal to zero. Therefore, the characteristic equation reduces to:

$$\sigma^2 - \mathrm{tr}(S)\,\sigma + \frac{1}{2}\left[\mathrm{tr}^2(S) - \mathrm{tr}(SS)\right] = 0 \Rightarrow$$

$$\sigma = \frac{\mathrm{tr}(S) \pm \sqrt{2\mathrm{tr}(SS) - \mathrm{tr}^2(S)}}{2}$$

9 Compressible

9.1 Cauchy stress tensor

The Cauchy stress tensor can be written in terms of the left Cauchy-Green deformation tensor as:

$$S = \frac{2\psi_1}{\hat{f}_3}\hat{B} + \frac{4\psi_2}{\hat{f}_3}\hat{B}\hat{B} + 2\psi_3\hat{f}_3 I$$

The traces of the first two powers of the Cauchy stress tensor, required to evaluate the coefficients of its characteristic equation, can be written as:

$$\mathrm{tr}(S) = \frac{2\psi_1}{\hat{f}_3}\hat{c}_1 + \frac{4\psi_2}{\hat{f}_3}\hat{c}_2 + 6\psi_3\hat{f}_3$$

$$\begin{aligned}
\mathrm{tr}(SS) = {} & \\
& +4\frac{\psi_1\psi_1}{\hat{c}_3}\hat{c}_2 + \\
& +8\frac{\psi_1\psi_2}{\hat{c}_3}\left(-\hat{c}_1^3 + 3\hat{c}_1\hat{c}_2 + 6\hat{c}_3\right) + \\
& +8\psi_1\psi_3\hat{c}_1 + \\
& +8\frac{\psi_2\psi_2}{\hat{c}_3}\left(-\hat{c}_1^4 + \hat{c}_2^2 + 2\hat{c}_1^2\hat{c}_2 + 8\hat{c}_1\hat{c}_3\right) + \\
& +16\psi_2\psi_3\hat{c}_2 + \\
& +12\psi_3\psi_3\hat{c}_3
\end{aligned}$$

9.2 Thickness

The derivative of the total potential energy with respect to the thickness of the element in the deformed state can be written as:

$$\phi = \sum \psi \frac{\alpha\delta}{2} - \omega \Rightarrow \frac{\partial\phi}{\partial\bar{\delta}} = \frac{\partial\psi}{\partial\bar{\delta}} \frac{\alpha\delta}{2}$$

$$\frac{\partial\phi}{\partial\bar{\delta}} = \left(\psi_1 \frac{\partial\hat{c}_1}{\partial\bar{\delta}} + \psi_2 \frac{\partial\hat{c}_2}{\partial\bar{\delta}} + \psi_3 \frac{\partial\hat{c}_3}{\partial\bar{\delta}}\right) \frac{\alpha\delta}{2}$$

However,

$$\frac{\partial\hat{c}_1}{\partial\bar{\delta}} = \frac{2}{\delta}\left(\frac{\bar{\delta}}{\delta}\right)$$

$$\frac{\partial\hat{c}_2}{\partial\bar{\delta}} = \frac{4}{\delta}\left(\frac{\bar{\delta}}{\delta}\right)^3$$

$$\frac{\partial\hat{c}_3}{\partial\bar{\delta}} = \frac{2\bar{\alpha}^2}{\delta\alpha^2}\left(\frac{\bar{\delta}}{\delta}\right)$$

Therefore,

$$\frac{\partial\phi}{\partial\bar{\delta}} = \alpha\left(\frac{\bar{\delta}}{\delta}\right)\left[\psi_1 + 2\psi_2\left(\frac{\bar{\delta}}{\delta}\right)^2 + \psi_3\left(\frac{\bar{\alpha}}{\alpha}\right)^2\right]$$

By setting the minimum of the total potential energy equal to zero, the following expression can be written:

$$\frac{\partial\phi}{\partial\bar{\delta}} = 0 \Rightarrow \psi_1 + 2\psi_2\left(\frac{\bar{\delta}}{\delta}\right)^2 + \psi_3\left(\frac{\bar{\alpha}}{\alpha}\right)^2 = 0$$

Notice that unconstrained minimization requires the following variable transformation to ensure a positive thickness in the deformed state.

$$\frac{\bar{\delta}}{\delta} = \tau^2 \Rightarrow \frac{\partial\tau}{\partial\bar{\delta}} = \frac{1}{2\delta\tau}$$

9.3 Stress orthogonal to the element’s surface

The traction vector related to the unit vector orthogonal to the element’s surface in the deformed state can be written as:

$$S\bar{w} = \frac{2\psi_1}{\hat{f}_3}\hat{B}\bar{w} + \frac{4\psi_2}{\hat{f}_3}\hat{B}\hat{B}\bar{w} + 2\psi_3\hat{f}_3\bar{w}$$

$$\hat{F}^T\bar{w} = \frac{\bar{\delta}}{\delta}w \Rightarrow \hat{B}\bar{w} = \left(\frac{\bar{\delta}}{\delta}\right)^2\bar{w} \Rightarrow$$

$$S\bar{w} = \left[\frac{2\psi_1}{\hat{f}_3}\left(\frac{\bar{\delta}}{\delta}\right)^2 + \frac{4\psi_2}{\hat{f}_3}\left(\frac{\bar{\delta}}{\delta}\right)^4 + 2\psi_3\hat{f}_3\right]\bar{w}$$

The unit vector $\bar{w}$ is a principal direction associated with a principal stress given by:

$$\sigma_3 = \frac{2\psi_1}{\hat{f}_3}\left(\frac{\bar{\delta}}{\delta}\right)^2 + \frac{4\psi_2}{\hat{f}_3}\left(\frac{\bar{\delta}}{\delta}\right)^4 + 2\psi_3\hat{f}_3$$

$$\hat{f}_3 = \frac{\bar{\delta}\bar{\alpha}}{\delta\alpha} \Rightarrow$$

$$\sigma_3 = 2\frac{\bar{\delta}\alpha}{\delta\bar{\alpha}}\left[\psi_1 + 2\psi_2\left(\frac{\bar{\delta}}{\delta}\right)^2 + \psi_3\left(\frac{\bar{\alpha}}{\alpha}\right)^2\right]$$

The minimum of the total potential energy implies that this principal stress is equal to zero.

$$\frac{\partial\phi}{\partial\bar{\delta}} = 0 \Rightarrow \sigma_3 = 0$$

9.4 Right Cauchy-Green deformation tensor

The invariants of the right Cauchy-Green deformation tensor and its derivatives with respect to the nodal displacements and the thickness of the element can be written as:

9.4.1 Invariant 1

$$\hat{c}_1 = \frac{1}{\alpha^2}\left[\left(\overline{v}^2\right)^T \overline{z}^1 - \left(\overline{v}^1\right)^T \overline{z}^2\right] + \tau^4$$

$$\frac{\partial \hat{c}_1}{\partial u^1} = \frac{2}{\alpha^2}\,\overline{z}^1$$

$$\frac{\partial \hat{c}_1}{\partial u^2} = \frac{2}{\alpha^2}\,\overline{z}^2$$

$$\frac{\partial \hat{c}_1}{\partial \tau} = 4\tau^3$$

9.4.2 Invariant 2

$$\begin{aligned}
\hat{c}_2 &= \\
&+ \frac{1}{\alpha^4}\left[\left(\overline{v}^1\right)^T \overline{v}^1 \left(\overline{z}^2\right)^T \overline{z}^2 - \left(\overline{v}^1\right)^T \overline{v}^2 \left(\overline{z}^2\right)^T \overline{z}^1\right] + \\
&+ \frac{1}{\alpha^4}\left[\left(\overline{v}^2\right)^T \overline{v}^2 \left(\overline{z}^1\right)^T \overline{z}^1 - \left(\overline{v}^1\right)^T \overline{v}^2 \left(\overline{z}^2\right)^T \overline{z}^1\right] + \\
&+ \tau^8
\end{aligned}$$

$$\frac{\partial \hat{c}_2}{\partial u^1} = \frac{4}{\alpha^4}\left[\left(\overline{z}^1\right)^T \overline{z}^1\overline{v}^2 - \left(\overline{z}^1\right)^T \overline{z}^2\overline{v}^1\right] = \frac{4}{\alpha^4}\left[\left(\overline{v}^2\right)^T \overline{z}^1\overline{z}^1 - \left(\overline{v}^1\right)^T \overline{z}^1\overline{z}^2\right]$$

$$\frac{\partial \hat{c}_2}{\partial u^2} = \frac{4}{\alpha^4}\left[\left(\overline{z}^2\right)^T \overline{z}^1\overline{v}^2 - \left(\overline{z}^2\right)^T \overline{z}^2\overline{v}^1\right] = \frac{4}{\alpha^4}\left[\left(\overline{v}^2\right)^T \overline{z}^2\overline{z}^1 - \left(\overline{v}^1\right)^T \overline{z}^2\overline{z}^2\right]$$

$$\frac{\partial \hat{c}_2}{\partial \tau} = 8\tau^7$$

9.4.3 Invariant 3

$$\hat{c}_3 = \left(\frac{\overline{\alpha}}{\alpha}\right)^2 \tau^4$$

$$\frac{\partial \hat{c}_3}{\partial u^1} = \frac{2\overline{\alpha}\tau^4}{\alpha^2}\left(\overline{w} \times \overline{v}^1\right)$$

$$\frac{\partial \hat{c}_3}{\partial u^2} = \frac{2\bar{\alpha}\tau^4}{\alpha^2}\left(\bar{w} \times \bar{v}^2\right)$$

$$\frac{\partial \hat{c}_3}{\partial \tau} = 4\left(\frac{\bar{\alpha}}{\alpha}\right)^2 \tau^3$$

9.5 Strain energy

The derivatives of the strain energy with respect to the nodal displacements and the thickness of the element can be written as:

$$\frac{\partial \psi}{\partial u^1}\,\frac{\alpha\delta}{2} = \left\{\frac{\psi_1}{\alpha}\,\bar{z}^1 + \frac{2\psi_2}{\alpha^3}\left[\left(\bar{z}^1\right)^T \bar{z}^1\bar{v}^2 - \left(\bar{z}^1\right)^T \bar{z}^2\bar{v}^1\right] + \psi_3\tau^4\left(\frac{\bar{\alpha}}{\alpha}\right)\left(\bar{w} \times \bar{v}^1\right)\right\}\delta$$

$$\frac{\partial \psi}{\partial u^2}\,\frac{\alpha\delta}{2} = \left\{\frac{\psi_1}{\alpha}\,\bar{z}^2 + \frac{2\psi_2}{\alpha^3}\left[\left(\bar{z}^2\right)^T \bar{z}^1\bar{v}^2 - \left(\bar{z}^2\right)^T \bar{z}^2\bar{v}^1\right] + \psi_3\tau^4\left(\frac{\bar{\alpha}}{\alpha}\right)\left(\bar{w} \times \bar{v}^2\right)\right\}\delta$$

$$\frac{\partial \psi}{\partial \bar{\delta}}\,\frac{\alpha\delta}{2} = \alpha\tau^2\left[\psi_1 + 2\psi_2\tau^4 + \psi_3\left(\frac{\bar{\alpha}}{\alpha}\right)^2\right]$$

9.6 Examples

Note that subtracting a constant from the strain energy function does not change the minimum points of the total potential energy. Therefore, only terms that depend on the invariants of the right Cauchy-Green deformation tensor are considered in the strain energy function.

Example 1: Consider a 10 x 10 square surface made of compressible Neo-Hookean material. The opposite vertical sides of the square surface are pulled apart with relative displacement equal to 3. The strain energy function for this material is given by:

$$\psi = \frac{\mu}{2}\,\hat{c}_1\left(\hat{c}_3\right)^{-\frac{1}{3}} + \frac{\kappa}{2}\left[\hat{c}_3 - 2\left(\hat{c}_3\right)^{\frac{1}{2}}\right]$$

$\mu = 0.4225$, $\kappa = 5.0000$

Figure 2 shows the meshes for the initial and final surfaces.

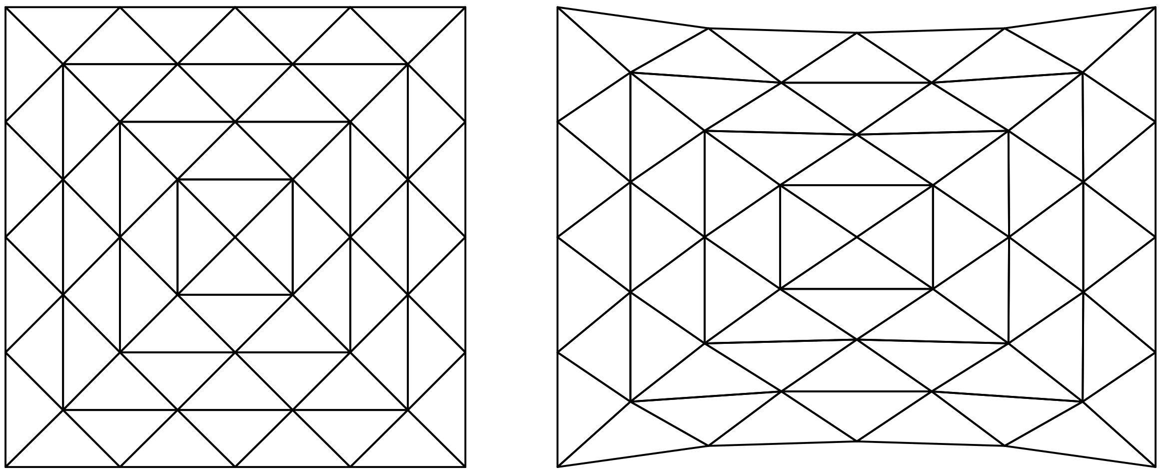

Figure 2

Table 1 shows the displacements in the Y direction of the bottom edge nodes compared with ANSYS.

Table 1

Node	Displ Y	ANSYS	Error (%)
2	0.46150	0.46150	0.00
3	0.55815	0.55815	0.00
4	0.46150	0.46150	0.00

Example 2: Consider a 10 x 10 square surface made of compressible Mooney-Rivlin material. The opposite vertical sides of the square surface are pulled apart with relative displacement equal to 3. The strain energy function for this material is given by:

$$\psi = \left[\mu_{10}\hat{c}_1 + \frac{1}{2}\mu_{01}\left(\hat{c}_1^2 - \hat{c}_2\right)\left(\hat{c}_3\right)^{-\frac{1}{3}} \right]\left(\hat{c}_3\right)^{-\frac{1}{3}} + \frac{\kappa}{2}\left[\hat{c}_3 - 2\left(\hat{c}_3\right)^{\frac{1}{2}} \right]$$

$$\mu_{10} = 0.3750 \ , \ \mu_{01} = -0.1250 \ , \ \kappa = 5.0000$$

Figure 3 shows the meshes for the initial and final surfaces.

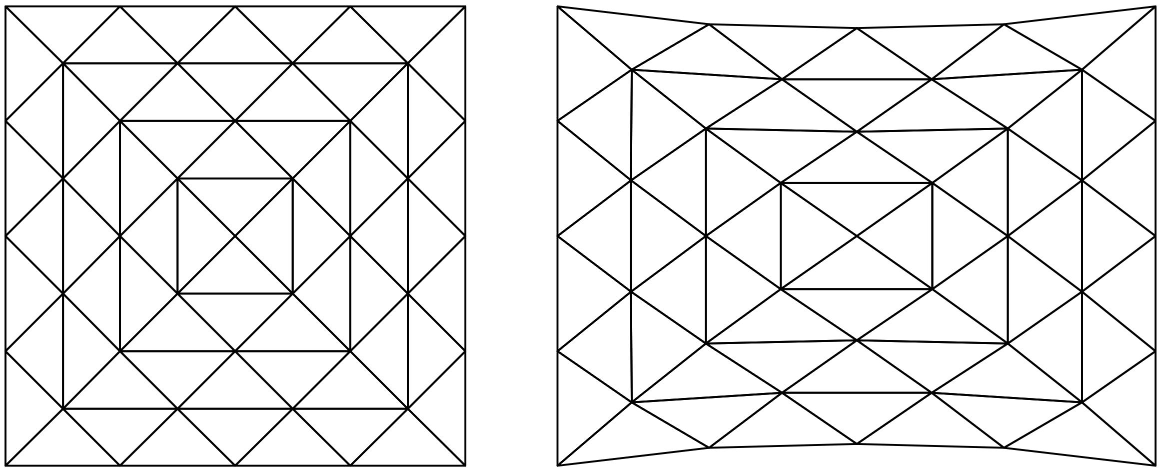

Figure 3

Table 2 shows the displacements in the Y direction of the bottom edge nodes compared with ANSYS.

Table 2

Node	Displ Y	ANSYS	Error (%)
2	0.39777	0.39777	0.00
3	0.48589	0.48589	0.00
4	0.39777	0.39777	0.00

Example 3: Consider an initially flat membrane made of compressible Neo-Hookean material. Two opposite corners are pulled downward while the other two opposite corners pulled upward with relative displacement equal to 2.5. The strain energy function for this material is given by:

$$\psi = \frac{\mu}{2}\,\hat{c}_1\left(\hat{c}_3\right)^{-\frac{1}{3}} + \frac{\kappa}{2}\left[\hat{c}_3 - 2\left(\hat{c}_3\right)^{\frac{1}{2}}\right]$$

$\mu = 208.33$, $\kappa = 277.78$

Figure 4 shows the meshes for the initial and final surfaces.

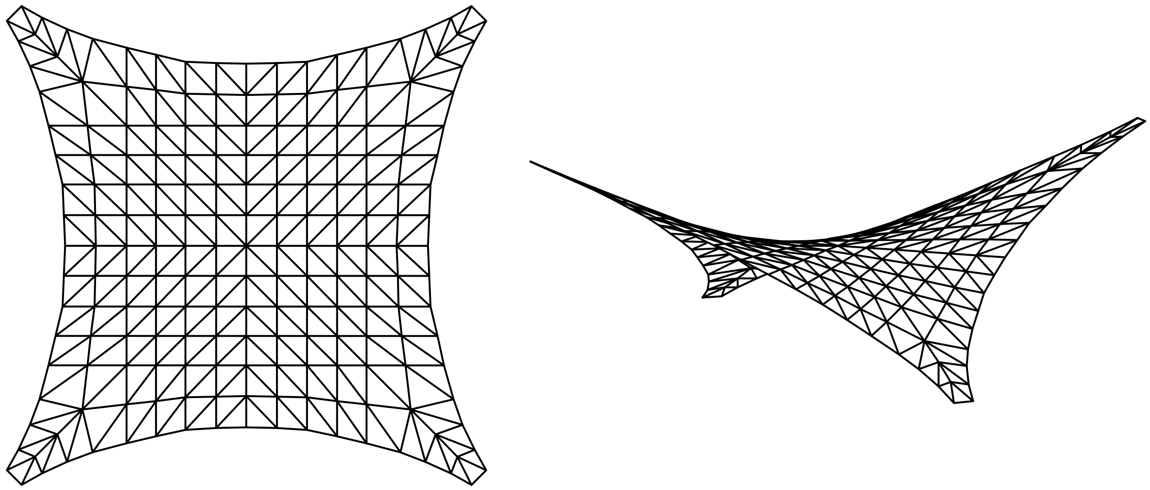

Figure 4

Table 3 shows the results for principal stress parallel to the element's surface.

Table 3

	Stress
Max	1.235857E+02
Min	-1.204032E+00

10 Incompressible

10.1 Cauchy stress tensor

The Cauchy stress tensor can be written in terms of the left Cauchy-Green deformation tensor as:

$$S = 2\psi_1\hat{B} + 4\psi_2\hat{B}\hat{B} - \gamma I$$

The traces of the first two powers of the Cauchy stress tensor, required to evaluate the coefficients of its characteristic equation, can be written as:

$$\mathrm{tr}\,(S) = 2\psi_1\hat{c}_1 + 4\psi_2\hat{c}_2 - 3\gamma$$

$$\mathrm{tr}\,(SS) =$$
$$+3\gamma^2 - 4\psi_1\gamma\hat{c}_1 - 8\psi_2\gamma\hat{c}_2 +$$
$$+4\psi_1\psi_1\hat{c}_2 +$$
$$+8\psi_1\psi_2\left(3\hat{c}_1\hat{c}_2 - \hat{c}_1^3 + 6\right) +$$
$$+8\psi_2\psi_2\left(2\hat{c}_1^2\hat{c}_2 + \hat{c}_2^2 - \hat{c}_1^4 + 8\hat{c}_1\right)$$

10.2 Thickness

By setting invariant 3 of the deformation gradient tensor equal to 1, the thickness of the element in the deformed state can be written as:

$$\hat{f}_3 = 1 \Rightarrow \frac{\overline{\delta}}{\delta} = \frac{\alpha}{\overline{\alpha}}$$

10.3 Stress orthogonal to the element's surface

The traction vector related to the unit vector orthogonal to the element's surface in the deformed state can be written as:

$$S\overline{w} = 2\psi_1\hat{B}\overline{w} + 4\psi_2\hat{B}\hat{B}\overline{w} - \gamma\overline{w}$$

$$\hat{F}^T\overline{w} = \frac{\overline{\delta}}{\delta}w \Rightarrow \hat{B}\overline{w} = \left(\frac{\overline{\delta}}{\delta}\right)^2\overline{w} = \left(\frac{\alpha}{\overline{\alpha}}\right)^2\overline{w} \Rightarrow$$

$$S\overline{w} = \left[2\psi_1\left(\frac{\alpha}{\overline{\alpha}}\right)^2 + 4\psi_2\left(\frac{\alpha}{\overline{\alpha}}\right)^4 - \gamma\right]\overline{w}$$

The unit vector $\overline{w}$ is a principal direction associated with a principal stress given by:

$$\sigma_3 = 2\psi_1\left(\frac{\alpha}{\overline{\alpha}}\right)^2 + 4\psi_2\left(\frac{\alpha}{\overline{\alpha}}\right)^4 - \gamma$$

The boundary condition implies that this stress is equal to zero. Therefore,

$$\gamma = 2\psi_1 \left(\frac{\alpha}{\bar{\alpha}}\right)^2 + 4\psi_2 \left(\frac{\alpha}{\bar{\alpha}}\right)^4 \Rightarrow$$

$$S = 2\psi_1 \hat{B} + 4\psi_2 \hat{B}\hat{B} - \left[2\psi_1 \left(\frac{\alpha}{\bar{\alpha}}\right)^2 + 4\psi_2 \left(\frac{\alpha}{\bar{\alpha}}\right)^4\right] I$$

10.4 Right Cauchy-Green deformation tensor

The invariants of the right Cauchy-Green deformation tensor and its derivatives with respect to the nodal displacements can be written as:

10.4.1 Invariant 1

$$\hat{c}_1 = \frac{1}{\alpha^2}\left[\left(\bar{v}^2\right)^T \bar{z}^1 - \left(\bar{v}^1\right)^T \bar{z}^2\right] + \left(\frac{\alpha}{\bar{\alpha}}\right)^2$$

$$\frac{\partial \hat{c}_1}{\partial u^1} = \frac{2}{\alpha^2}\bar{z}^1 - \frac{2\alpha^2}{\bar{\alpha}^3}\left(\bar{w} \times \bar{v}^1\right)$$

$$\frac{\partial \hat{c}_1}{\partial u^2} = \frac{2}{\alpha^2}\bar{z}^2 - \frac{2\alpha^2}{\bar{\alpha}^3}\left(\bar{w} \times \bar{v}^2\right)$$

10.4.2 Invariant 2

$$\begin{aligned}
\hat{c}_2 ={}& \\
&+ \frac{1}{\alpha^4}\left[\left(\bar{v}^1\right)^T \bar{v}^1 \left(\bar{z}^2\right)^T \bar{z}^2 - \left(\bar{v}^1\right)^T \bar{v}^2 \left(\bar{z}^2\right)^T \bar{z}^1\right] + \\
&+ \frac{1}{\alpha^4}\left[\left(\bar{v}^2\right)^T \bar{v}^2 \left(\bar{z}^1\right)^T \bar{z}^1 - \left(\bar{v}^1\right)^T \bar{v}^2 \left(\bar{z}^2\right)^T \bar{z}^1\right] + \\
&+ \left(\frac{\alpha}{\bar{\alpha}}\right)^4
\end{aligned}$$

$$\frac{\partial \hat{c}_2}{\partial u^1} =$$

$$+\frac{4}{\alpha^4}\left(\bar{z}^1\right)^T \bar{z}^1\bar{v}^2 - \frac{4}{\alpha^4}\left(\bar{z}^1\right)^T \bar{z}^2\bar{v}^1 - \frac{4\alpha^4}{\bar{\alpha}^5}\left(\bar{w} \times \bar{v}^1\right) =$$

$$+\frac{4}{\alpha^4}\left(\bar{v}^2\right)^T \bar{z}^1\bar{z}^1 - \frac{4}{\alpha^4}\left(\bar{v}^1\right)^T \bar{z}^1\bar{z}^2 - \frac{4\alpha^4}{\bar{\alpha}^5}\left(\bar{w} \times \bar{v}^1\right)$$

$$\frac{\partial \hat{c}_2}{\partial u^2} =$$

$$+\frac{4}{\alpha^4}\left(\bar{z}^2\right)^T \bar{z}^1\bar{v}^2 - \frac{4}{\alpha^4}\left(\bar{z}^2\right)^T \bar{z}^2\bar{v}^1 - \frac{4\alpha^4}{\bar{\alpha}^5}\left(\bar{w} \times \bar{v}^2\right) =$$

$$+\frac{4}{\alpha^4}\left(\bar{v}^2\right)^T \bar{z}^2\bar{z}^1 - \frac{4}{\alpha^4}\left(\bar{v}^1\right)^T \bar{z}^2\bar{z}^2 - \frac{4\alpha^4}{\bar{\alpha}^5}\left(\bar{w} \times \bar{v}^2\right)$$

10.5 Strain energy

The derivatives of the strain energy with respect to the nodal displacements of the element can be written as:

$$\frac{\partial \psi}{\partial u^1}\,\frac{\alpha\delta}{2} =$$

$$+\psi_1\left[\frac{1}{\alpha}\,\bar{z}^1 - \left(\frac{\alpha}{\bar{\alpha}}\right)^3\left(\bar{w} \times \bar{v}^1\right)\right]\delta \,+$$

$$+2\psi_2\left[\frac{1}{\alpha^3}\left(\bar{z}^1\right)^T \bar{z}^1\bar{v}^2 - \frac{1}{\alpha^3}\left(\bar{z}^1\right)^T \bar{z}^2\bar{v}^1 - \left(\frac{\alpha}{\bar{\alpha}}\right)^5\left(\bar{w} \times \bar{v}^1\right)\right]\delta$$

$$\frac{\partial \psi}{\partial u^2}\,\frac{\alpha\delta}{2} =$$

$$+\psi_1\left[\frac{1}{\alpha}\,\bar{z}^2 - \left(\frac{\alpha}{\bar{\alpha}}\right)^3\left(\bar{w} \times \bar{v}^2\right)\right]\delta \,+$$

$$+2\psi_2\left[\frac{1}{\alpha^3}\left(\bar{z}^2\right)^T \bar{z}^1\bar{v}^2 - \frac{1}{\alpha^3}\left(\bar{z}^2\right)^T \bar{z}^2\bar{v}^1 - \left(\frac{\alpha}{\bar{\alpha}}\right)^5\left(\bar{w} \times \bar{v}^2\right)\right]\delta$$

10.6 Examples

Note that subtracting a constant from the strain energy function does not change the minimum points of the total potential energy. Therefore, only terms that depend on the invariants of the right Cauchy-Green deformation tensor are considered in the strain energy function.

Example 1: Consider a 10 x 10 square surface made of incompressible Neo-Hookean material. The opposite vertical sides of the square surface are pulled apart with relative displacement equal to 3. The strain energy function for this material is given by:

$$\psi = \frac{\mu}{2}\hat{c}_1$$

$$\mu = 0.4225$$

Figure 5 shows the meshes for the initial and final surfaces.

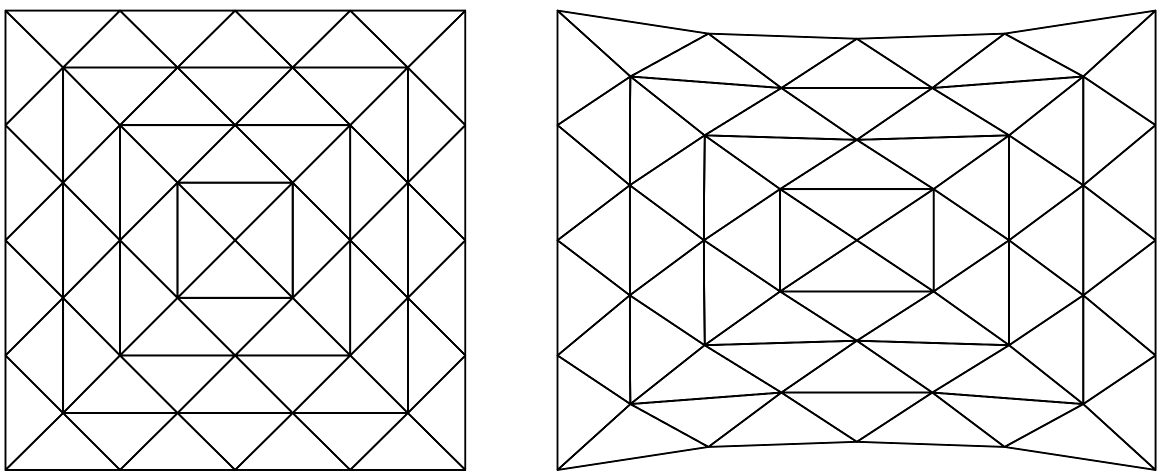

Figure 5

Table 4 shows the displacements in the Y direction of the bottom edge nodes compared with ANSYS.

Table 4

Node	Displ Y	ANSYS	Error (%)
2	0.50989	0.50989	0.00
3	0.61739	0.61739	0.00
4	0.50989	0.50989	0.00

Example 2: Consider a 10 x 10 square surface made of incompressible Mooney-Rivlin material. The opposite vertical sides of the square surface are pulled apart with relative displacement equal to 3. The strain energy function for this material is given by:

$$\psi = \mu_{10}\hat{c}_1 + \frac{1}{2}\mu_{01}\left(\hat{c}_1^2 - \hat{c}_2\right)$$

$$\mu_{10} = 0.3750 \ , \ \mu_{01} = -0.1250$$

Figure 6 shows the meshes for the initial and final surfaces.

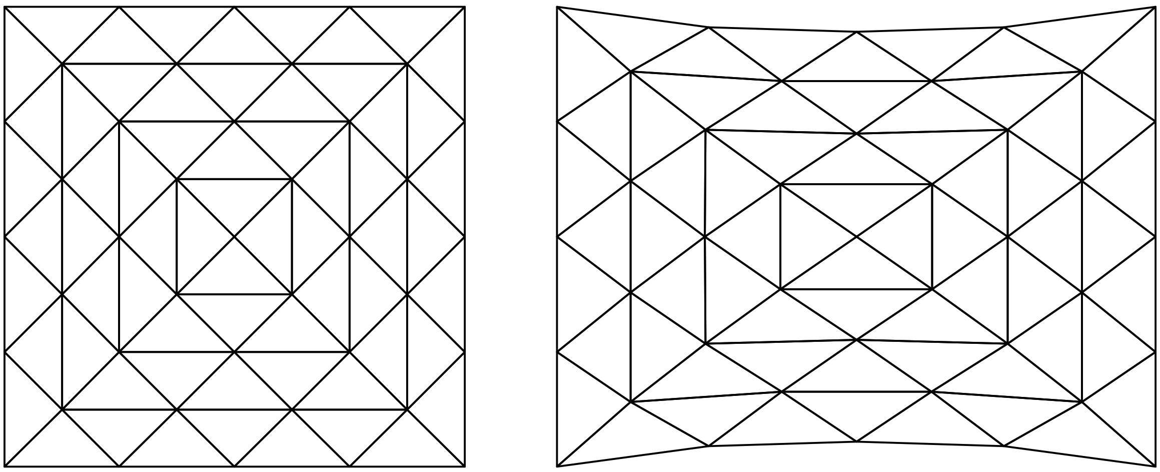

Figure 6

Table 5 shows the displacements in the Y direction of the bottom edge nodes compared with ANSYS.

Table 5

Node	Displ Y	ANSYS	Error (%)
2	0.44601	0.44601	0.00
3	0.54504	0.54504	0.00
4	0.44601	0.44601	0.00

Example 3: Consider an initially flat membrane made of incompressible Neo-Hookean material. Two opposite corners are pulled downward while the other two opposite corners pulled upward with relative displacement equal to 2.5. The strain energy function for this material is given by:

$$\psi = \frac{\mu}{2}\hat{c}_1$$

$$\mu = 208.33$$

Figure 7 shows the meshes for the initial and final surfaces.

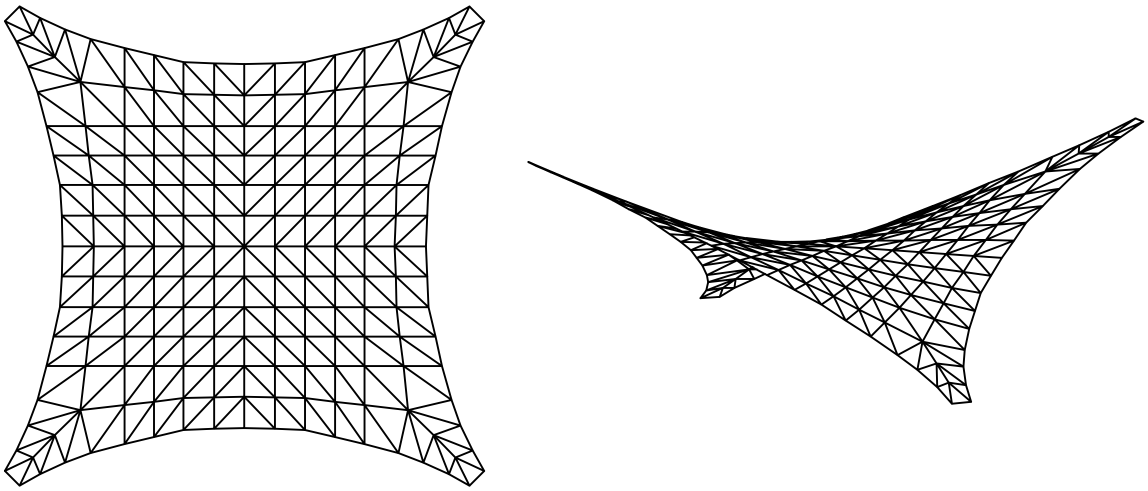

Figure 7

Table 6 shows the results for principal stress parallel to the element’s surface.

Table 6

	Stress
Max	1.842568E+02
Min	-6.711738E+00

Example 4: Consider a 10 x 10 square surface made of incompressible Arruda-Boyce material. The opposite vertical sides of the square surface are pulled apart with relative displacement equal to 3. The Taylor series expansion of the strain energy function for this material is given by:

$$\psi = \mu\left(\frac{1}{2}\hat{c}_1 + \frac{1}{20\lambda^2}\hat{c}_1^2 + \frac{11}{1050\lambda^4}\hat{c}_1^3 + \frac{19}{7000\lambda^6}\hat{c}_1^4 + \frac{519}{673750\lambda^8}\hat{c}_1^5\right)$$

$\mu = 0.1$, $\lambda = 1.05$

Figure 8 shows the meshes for the initial and final surfaces.

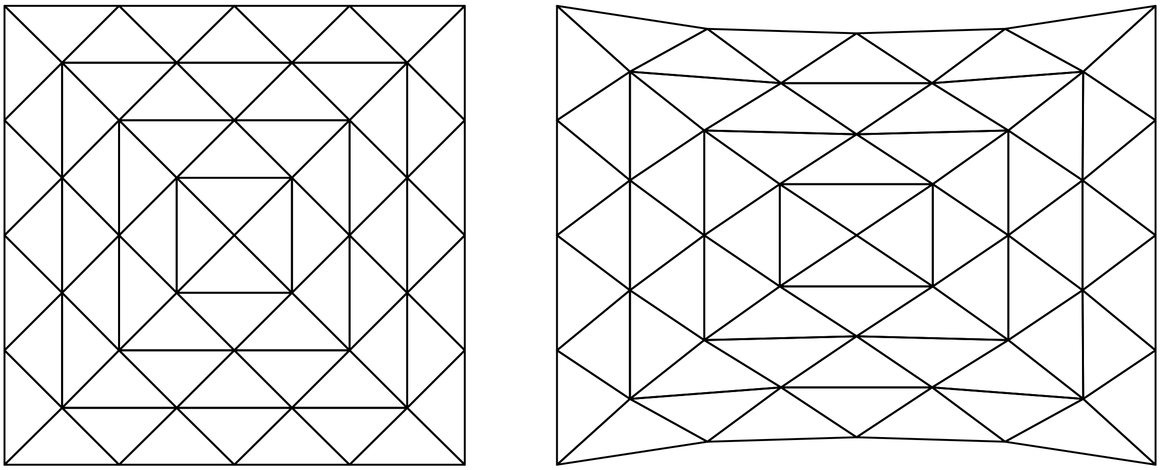

Figure 8

Table 7 shows the displacements in the Y direction of the bottom edge nodes compared with ANSYS.

Table 7

Node	Displ Y	ANSYS	Error (%)
2	0.50491	0.50648	-0.31
3	0.61113	0.61304	-0.31
4	0.50491	0.50648	-0.31

Considering ℓ^{-1} as the inverse Langevin function, the strain energy function for the Arruda-Boyce material is given by:

$$\psi = \mu\lambda\sqrt{\frac{\hat{c}_1}{3}}\,\ell^{-1}\left(\frac{1}{\lambda}\sqrt{\frac{\hat{c}_1}{3}}\right) + \mu\lambda^2 \ln\left\{\frac{\ell^{-1}\left(\frac{1}{\lambda}\sqrt{\frac{\hat{c}_1}{3}}\right)}{\sinh\left[\ell^{-1}\left(\frac{1}{\lambda}\sqrt{\frac{\hat{c}_1}{3}}\right)\right]}\right\}$$

11 Appendix - ANSYS

In other texts, including ANSYS manual, the following definitions of the invariants are frequently used.

$$J = \det\left(\hat{F}\right) = \left(\hat{c}_3\right)^{\frac{1}{2}}$$

$$\bar{F} = J^{-\frac{1}{3}}\hat{F} \Rightarrow \det\left(\bar{F}\right) = 1$$

$$\overline{C} = \overline{F}^T\overline{F} \Rightarrow \overline{C} = J^{-\frac{2}{3}}\hat{C}$$

$$\overline{I}_1 = tr\left(\overline{C}\right) = J^{-\frac{2}{3}}\hat{c}_1 = \hat{c}_1\left(\hat{c}_3\right)^{-\frac{1}{3}}$$

$$\overline{I}_2 = \frac{1}{2}\left[tr^2\left(\overline{C}\right) - tr\left(\overline{C}^T\overline{C}\right)\right] = \frac{1}{2}J^{-\frac{4}{3}}\left(\hat{c}_1^2 - \hat{c}_2\right) = \frac{1}{2}\left(\hat{c}_1^2 - \hat{c}_2\right)\left(\hat{c}_3\right)^{-\frac{2}{3}}$$

$$\overline{I}_3 = det\left(\overline{C}\right) = J^{-2}\hat{c}_3 = 1$$