

The deformation gradient and the right Cauchy-Green deformation tensor for triangle element

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1 Geometry

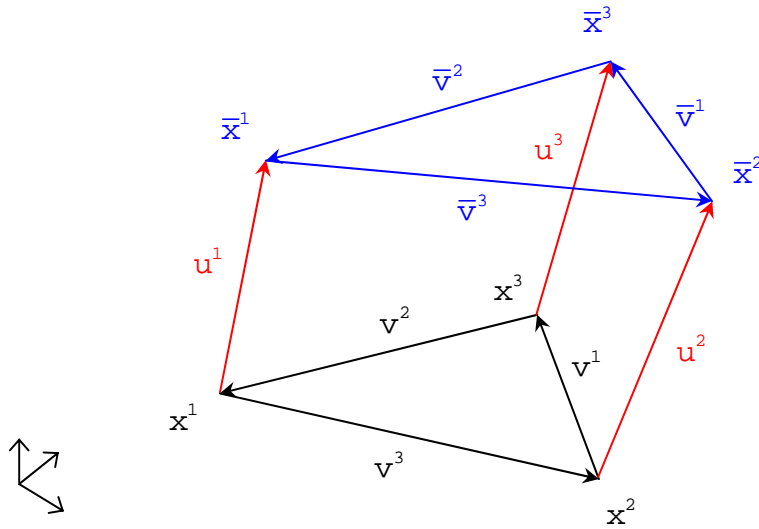


Figure 1

$$\bar{v}^1 = v^1 + u^3 - u^2$$

$$\bar{v}^2 = v^2 + u^1 - u^3$$

$$\bar{v}^3 = v^3 + u^2 - u^1$$

$$w = \frac{v^1 \times v^2}{\|v^1 \times v^2\|}$$

$$\alpha = w^T (v^1 \times v^2)$$

$$\bar{w} = \frac{\bar{v}^1 \times \bar{v}^2}{\|\bar{v}^1 \times \bar{v}^2\|}$$

$$\bar{\alpha} = \bar{w}^T (\bar{v}^1 \times \bar{v}^2)$$

$$w^i = w \times v^i, \quad i = 1, 2, 3$$

The unit vectors w and \bar{w} are orthogonal to the element's surface in the undeformed state and deformed state respectively. Notice that these vectors point toward the observer, when nodes associated with the element appear counterclockwise.

The scalars α and $\bar{\alpha}$ are equal to twice the area of the element in the undeformed state and deformed state respectively.

The scalars δ and $\bar{\delta}$ are equal to the thickness of the element in the undeformed state and deformed state respectively.

2 Convex combination of vertexes

The convex combination of vertexes in the deformed state is given by:

$$\bar{\mathbf{x}} = \alpha_1 \bar{\mathbf{x}}^1 + \alpha_2 \bar{\mathbf{x}}^2 + \alpha_3 \bar{\mathbf{x}}^3$$

$$\alpha_1 + \alpha_2 + \alpha_3 = 1$$

$$\alpha_i \geq 0, \quad i = 1, 2, 3$$

However,

$$\left. \begin{aligned} \bar{\mathbf{x}}^1 &= \mathbf{u}^1 + \mathbf{x}^1 \\ \bar{\mathbf{x}}^2 &= \mathbf{u}^2 + \mathbf{x}^2 \\ \bar{\mathbf{x}}^3 &= \mathbf{u}^3 + \mathbf{x}^3 \end{aligned} \right\} \Rightarrow$$

$$\bar{\mathbf{x}} = \alpha_1 \mathbf{u}^1 + \alpha_2 \mathbf{u}^2 + \alpha_3 \mathbf{u}^3 + (\alpha_1 \mathbf{x}^1 + \alpha_2 \mathbf{x}^2 + \alpha_3 \mathbf{x}^3)$$

The convex combination of vertexes in the undeformed state is the point \mathbf{x} . Therefore,

$$\mathbf{x} = \alpha_1 \mathbf{x}^1 + \alpha_2 \mathbf{x}^2 + \alpha_3 \mathbf{x}^3 \Rightarrow$$

$$\bar{\mathbf{x}} = \alpha_1 \mathbf{u}^1 + \alpha_2 \mathbf{u}^2 + \alpha_3 \mathbf{u}^3 + \mathbf{x}$$

2.1 Alpha coefficients

$$\mathbf{x} = \alpha_1 \mathbf{x}^1 + \alpha_2 \mathbf{x}^2 + \alpha_3 \mathbf{x}^3$$

$$\alpha_1 + \alpha_2 + \alpha_3 = 1 \Rightarrow$$

$$\alpha_1 (\mathbf{x}^1 - \mathbf{x}^3) + \alpha_2 (\mathbf{x}^2 - \mathbf{x}^3) = \mathbf{x} - \mathbf{x}^3$$

$$\left. \begin{aligned} \mathbf{v}^1 &= \mathbf{x}^3 - \mathbf{x}^2 \\ \mathbf{v}^2 &= \mathbf{x}^1 - \mathbf{x}^3 \end{aligned} \right\} \Rightarrow$$

$$\alpha_1 \mathbf{v}^2 - \alpha_2 \mathbf{v}^1 = \mathbf{x} - \mathbf{x}^3$$

2.1.1 Coefficient alpha 1

$$(\mathbf{w}^1)^T (\alpha_1 \mathbf{v}^2 - \alpha_2 \mathbf{v}^1) = (\mathbf{w}^1)^T (\mathbf{x} - \mathbf{x}^3)$$

$$\alpha_1 = \frac{(\mathbf{w}^1)^T (\mathbf{x} - \mathbf{x}^3)}{\mathbf{w}^T (\mathbf{v}^1 \times \mathbf{v}^2)} = \frac{(\mathbf{w}^1)^T (\mathbf{x} - \mathbf{x}^3)}{\alpha}$$

2.1.2 Coefficient alpha 2

$$(\mathbf{w}^2)^T (\alpha_1 \mathbf{v}^2 - \alpha_2 \mathbf{v}^1) = (\mathbf{w}^2)^T (\mathbf{x} - \mathbf{x}^3)$$

$$\alpha_2 = \frac{(\mathbf{w}^2)^T (\mathbf{x} - \mathbf{x}^3)}{\mathbf{w}^T (\mathbf{v}^1 \times \mathbf{v}^2)} = \frac{(\mathbf{w}^2)^T (\mathbf{x} - \mathbf{x}^3)}{\alpha}$$

2.1.3 Geometric interpretation

Each coefficient alpha can be interpreted as the division of a fraction of the undeformed area by the total undeformed area.

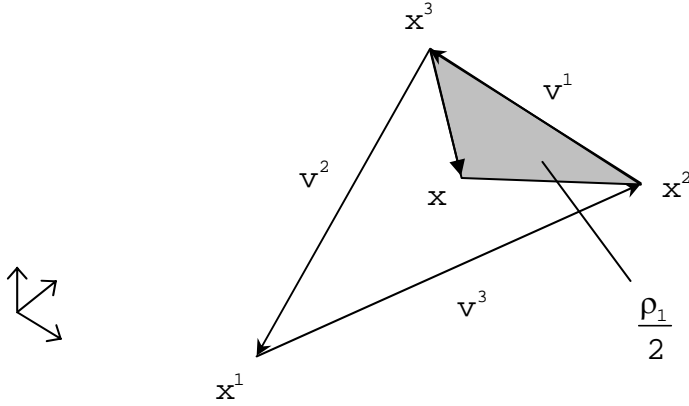


Figure 2

$$\rho_1 w = (x - x^3) \times (-v^1) \Rightarrow \rho_1 = (w^1)^T (x - x^3)$$

$$\alpha_1 = \frac{(w^1)^T (x - x^3)}{\alpha} = \frac{\rho_1}{\alpha} \Rightarrow 0 \leq \alpha_1 \leq 1$$

Notice that the non-negative constraint for α_1 is satisfied. Corresponding expressions can be written for the other coefficients.

3 Plane strain

It is important to emphasize that $\bar{w} = w$ for the plane strain case. However, unless explicitly indicated, these two vectors will be treated as different vectors. In this way, the expressions for the plane strain case can be used as part of the expressions for the plane stress case.

3.1 Deformation gradient tensor

$$\bar{x} = \alpha_1 (u^1 - u^3) + \alpha_2 (u^2 - u^3) + u^3 + x$$

$$f_{ij} = \frac{\partial \bar{x}_i}{\partial x_j}$$

$$f_{ij} = \delta_{ij} + \frac{(u^1 - u^3)_i w_j^1}{\alpha} + \frac{(u^2 - u^3)_i w_j^2}{\alpha}$$

$$d^i = u^i - u^3 \Rightarrow$$

$$f_{ij} = \delta_{ij} + \frac{d_i^1 w_j^1}{\alpha} + \frac{d_i^2 w_j^2}{\alpha}$$

$$F = I + \frac{1}{\alpha} \left[d^1 (w^1)^T + d^2 (w^2)^T \right]$$

$$d^i = u^i - u^3 \Rightarrow$$

$$F = I + \frac{1}{\alpha} \left[u^1 (w^1)^T + u^2 (w^2)^T - u^3 (w^1 + w^2)^T \right]$$

$$w^1 + w^2 + w^3 = 0 \Rightarrow$$

$$F = I + \frac{1}{\alpha} \left[u^1 (w^1)^T + u^2 (w^2)^T + u^3 (w^3)^T \right]$$

Notice that,

$$Fv^1 = v^1 + u^3 - u^2 = \bar{v}^1$$

$$Fv^2 = v^2 + u^1 - u^3 = \bar{v}^2$$

$$Fv^3 = v^3 + u^2 - u^1 = \bar{v}^3$$

$$Fw = w$$

3.1.1 Invariant 1

$$f_1 = \text{tr}(F)$$

$$F = I + \frac{1}{\alpha} \left[d^1 (w^1)^T + d^2 (w^2)^T \right]$$

$$f_1 = 3 + \frac{1}{\alpha} \left[(w^1)^T d^1 + (w^2)^T d^2 \right]$$

$$\left. \begin{aligned} d^1 &= \bar{v}^2 - v^2 \\ d^2 &= v^1 - \bar{v}^1 \end{aligned} \right\} \Rightarrow$$

$$f_1 = 3 + \frac{1}{\alpha} \left[(w^2)^T v^1 - (w^1)^T v^2 + (w^1)^T \bar{v}^2 - (w^2)^T \bar{v}^1 \right]$$

$$\mathbf{w}^i = \mathbf{w} \times \mathbf{v}^i \Rightarrow$$

$$f_1 = 3 + \frac{1}{\alpha} \left[-2\mathbf{w}^T (\mathbf{v}^1 \times \mathbf{v}^2) + (\mathbf{w}^1)^T \bar{\mathbf{v}}^2 - (\mathbf{w}^2)^T \bar{\mathbf{v}}^1 \right]$$

$$\alpha = \mathbf{w}^T (\mathbf{v}^1 \times \mathbf{v}^2) \Rightarrow$$

$$f_1 = 1 + \frac{1}{\alpha} \left[(\mathbf{w}^1)^T \bar{\mathbf{v}}^2 - (\mathbf{w}^2)^T \bar{\mathbf{v}}^1 \right]$$

3.1.2 Invariant 2

$$f_2 = \text{tr} (\mathbf{F}^T \mathbf{F})$$

$$\mathbf{F} = \mathbf{I} + \frac{1}{\alpha} \left[\mathbf{d}^1 (\mathbf{w}^1)^T + \mathbf{d}^2 (\mathbf{w}^2)^T \right]$$

$$\begin{aligned} (\mathbf{F}^T \mathbf{F} - \mathbf{I}) \alpha^2 = & \\ & + \alpha \mathbf{d}^1 (\mathbf{w}^1)^T + \\ & + \alpha \mathbf{d}^2 (\mathbf{w}^2)^T + \\ & + \alpha \mathbf{w}^1 (\mathbf{d}^1)^T + \\ & + \alpha \mathbf{w}^2 (\mathbf{d}^2)^T + \\ & + (\mathbf{d}^1)^T \mathbf{d}^1 \mathbf{w}^1 (\mathbf{w}^1)^T + \\ & + (\mathbf{d}^1)^T \mathbf{d}^2 \mathbf{w}^1 (\mathbf{w}^2)^T + \\ & + (\mathbf{d}^2)^T \mathbf{d}^1 \mathbf{w}^2 (\mathbf{w}^1)^T + \\ & + (\mathbf{d}^2)^T \mathbf{d}^2 \mathbf{w}^2 (\mathbf{w}^2)^T \end{aligned}$$

$$\begin{aligned}
& (\mathbf{f}_2 - 3) \alpha^2 = \\
& +2\alpha \left(\mathbf{w}^1 \right)^T \mathbf{d}^1 + \\
& +2\alpha \left(\mathbf{w}^2 \right)^T \mathbf{d}^2 + \\
& + \left(\mathbf{d}^1 \right)^T \mathbf{d}^1 \left(\mathbf{w}^1 \right)^T \mathbf{w}^1 + \\
& +2 \left(\mathbf{d}^1 \right)^T \mathbf{d}^2 \left(\mathbf{w}^2 \right)^T \mathbf{w}^1 + \\
& + \left(\mathbf{d}^2 \right)^T \mathbf{d}^2 \left(\mathbf{w}^2 \right)^T \mathbf{w}^2
\end{aligned}$$

$$\mathbf{w}^i = \mathbf{w} \times \mathbf{v}^i \Rightarrow$$

$$\begin{aligned}
& (\mathbf{f}_2 - 3) \alpha^2 = \\
& +2\alpha \left(\mathbf{d}^1 \right)^T \left(\mathbf{w} \times \mathbf{v}^1 \right) + \\
& +2\alpha \left(\mathbf{d}^2 \right)^T \left(\mathbf{w} \times \mathbf{v}^2 \right) + \\
& + \left(\mathbf{d}^1 \right)^T \mathbf{d}^1 \left(\mathbf{v}^1 \right)^T \mathbf{v}^1 + \\
& +2 \left(\mathbf{d}^1 \right)^T \mathbf{d}^2 \left(\mathbf{v}^2 \right)^T \mathbf{v}^1 + \\
& + \left(\mathbf{d}^2 \right)^T \mathbf{d}^2 \left(\mathbf{v}^2 \right)^T \mathbf{v}^2
\end{aligned}$$

$$\left. \begin{aligned} \mathbf{d}^1 &= \bar{\mathbf{v}}^2 - \mathbf{v}^2 \\ \mathbf{d}^2 &= \mathbf{v}^1 - \bar{\mathbf{v}}^1 \end{aligned} \right\} \Rightarrow$$

$$\begin{aligned}
& (\mathbf{f}_2 - 3) \alpha^2 = \\
& -4\alpha \mathbf{w}^T \left(\mathbf{v}^1 \times \mathbf{v}^2 \right) + \\
& +2 \left(\mathbf{v}^1 \right)^T \left[\mathbf{v}^2 \times \left(\mathbf{v}^1 \times \mathbf{v}^2 \right) \right] + \\
& +2 \left(\bar{\mathbf{v}}^2 \right)^T \left[\mathbf{v}^1 \times \left(\mathbf{v}^1 \times \mathbf{v}^2 \right) + \alpha \left(\mathbf{w} \times \mathbf{v}^1 \right) \right] + \\
& -2 \left(\bar{\mathbf{v}}^1 \right)^T \left[\mathbf{v}^2 \times \left(\mathbf{v}^1 \times \mathbf{v}^2 \right) + \alpha \left(\mathbf{w} \times \mathbf{v}^2 \right) \right] + \\
& + \left(\bar{\mathbf{v}}^1 \right)^T \bar{\mathbf{v}}^1 \left(\mathbf{v}^2 \right)^T \mathbf{v}^2 + \left(\bar{\mathbf{v}}^2 \right)^T \bar{\mathbf{v}}^2 \left(\mathbf{v}^1 \right)^T \mathbf{v}^1 - 2 \left(\bar{\mathbf{v}}^2 \right)^T \bar{\mathbf{v}}^1 \left(\mathbf{v}^2 \right)^T \mathbf{v}^1
\end{aligned}$$

$$\alpha \mathbf{w} = \mathbf{v}^1 \times \mathbf{v}^2 \Rightarrow$$

$$\mathbf{f}_2 = 1 + \frac{1}{\alpha^2} \left[\left(\bar{\mathbf{v}}^2 \right)^T \bar{\mathbf{z}}^1 - \left(\bar{\mathbf{v}}^1 \right)^T \bar{\mathbf{z}}^2 \right]$$

Where,

$$\overline{z}^1 = \left[\left(v^1 \right)^T v^1 \overline{v}^2 - \left(v^1 \right)^T v^2 \overline{v}^1 \right]$$

$$\overline{z}^2 = \left[\left(v^2 \right)^T v^1 \overline{v}^2 - \left(v^2 \right)^T v^2 \overline{v}^1 \right]$$

3.1.3 Invariant 3

$$f_{ij} = \delta_{ij} + \frac{1}{\alpha} \left(d_i^1 w_j^1 + d_i^2 w_j^2 \right)$$

$$f_3 = \det \left(F \right) = f_{11} f_{22} f_{33} + f_{12} f_{23} f_{31} + f_{13} f_{32} f_{21} - f_{31} f_{22} f_{13} - f_{32} f_{23} f_{11} - f_{33} f_{12} f_{21}$$

$$\begin{aligned}
& (\mathfrak{f}_3 - 1) \alpha = \\
& + \left(\mathfrak{d}_1^1 \mathfrak{w}_1^1 + \mathfrak{d}_1^2 \mathfrak{w}_1^2 \right) + \\
& + \left(\mathfrak{d}_2^1 \mathfrak{w}_2^1 + \mathfrak{d}_2^2 \mathfrak{w}_2^2 \right) + \\
& + \left(\mathfrak{d}_3^1 \mathfrak{w}_3^1 + \mathfrak{d}_3^2 \mathfrak{w}_3^2 \right) + \\
& + \frac{1}{\alpha} \left(\mathfrak{d}_1^1 \mathfrak{w}_1^1 + \mathfrak{d}_1^2 \mathfrak{w}_1^2 \right) \left(\mathfrak{d}_2^1 \mathfrak{w}_2^1 + \mathfrak{d}_2^2 \mathfrak{w}_2^2 \right) + \\
& - \frac{1}{\alpha} \left(\mathfrak{d}_1^1 \mathfrak{w}_2^1 + \mathfrak{d}_1^2 \mathfrak{w}_2^2 \right) \left(\mathfrak{d}_2^1 \mathfrak{w}_1^1 + \mathfrak{d}_2^2 \mathfrak{w}_1^2 \right) + \\
& + \frac{1}{\alpha} \left(\mathfrak{d}_1^1 \mathfrak{w}_1^1 + \mathfrak{d}_1^2 \mathfrak{w}_1^2 \right) \left(\mathfrak{d}_3^1 \mathfrak{w}_3^1 + \mathfrak{d}_3^2 \mathfrak{w}_3^2 \right) + \\
& - \frac{1}{\alpha} \left(\mathfrak{d}_1^1 \mathfrak{w}_3^1 + \mathfrak{d}_1^2 \mathfrak{w}_3^2 \right) \left(\mathfrak{d}_3^1 \mathfrak{w}_1^1 + \mathfrak{d}_3^2 \mathfrak{w}_1^2 \right) + \\
& + \frac{1}{\alpha} \left(\mathfrak{d}_2^1 \mathfrak{w}_2^1 + \mathfrak{d}_2^2 \mathfrak{w}_2^2 \right) \left(\mathfrak{d}_3^1 \mathfrak{w}_3^1 + \mathfrak{d}_3^2 \mathfrak{w}_3^2 \right) + \\
& - \frac{1}{\alpha} \left(\mathfrak{d}_2^1 \mathfrak{w}_3^1 + \mathfrak{d}_2^2 \mathfrak{w}_3^2 \right) \left(\mathfrak{d}_3^1 \mathfrak{w}_2^1 + \mathfrak{d}_3^2 \mathfrak{w}_2^2 \right) + \\
& + \frac{1}{\alpha^2} \left(\mathfrak{d}_1^1 \mathfrak{w}_1^1 + \mathfrak{d}_1^2 \mathfrak{w}_1^2 \right) \left(\mathfrak{d}_2^1 \mathfrak{w}_2^1 + \mathfrak{d}_2^2 \mathfrak{w}_2^2 \right) \left(\mathfrak{d}_3^1 \mathfrak{w}_3^1 + \mathfrak{d}_3^2 \mathfrak{w}_3^2 \right) + \\
& + \frac{1}{\alpha^2} \left(\mathfrak{d}_1^1 \mathfrak{w}_2^1 + \mathfrak{d}_1^2 \mathfrak{w}_2^2 \right) \left(\mathfrak{d}_2^1 \mathfrak{w}_3^1 + \mathfrak{d}_2^2 \mathfrak{w}_3^2 \right) \left(\mathfrak{d}_3^1 \mathfrak{w}_1^1 + \mathfrak{d}_3^2 \mathfrak{w}_1^2 \right) + \\
& + \frac{1}{\alpha^2} \left(\mathfrak{d}_1^1 \mathfrak{w}_3^1 + \mathfrak{d}_1^2 \mathfrak{w}_3^2 \right) \left(\mathfrak{d}_3^1 \mathfrak{w}_2^1 + \mathfrak{d}_3^2 \mathfrak{w}_2^2 \right) \left(\mathfrak{d}_2^1 \mathfrak{w}_1^1 + \mathfrak{d}_2^2 \mathfrak{w}_1^2 \right) + \\
& - \frac{1}{\alpha^2} \left(\mathfrak{d}_1^1 \mathfrak{w}_1^1 + \mathfrak{d}_1^2 \mathfrak{w}_1^2 \right) \left(\mathfrak{d}_2^1 \mathfrak{w}_3^1 + \mathfrak{d}_2^2 \mathfrak{w}_3^2 \right) \left(\mathfrak{d}_3^1 \mathfrak{w}_2^1 + \mathfrak{d}_3^2 \mathfrak{w}_2^2 \right) + \\
& - \frac{1}{\alpha^2} \left(\mathfrak{d}_2^1 \mathfrak{w}_2^1 + \mathfrak{d}_2^2 \mathfrak{w}_2^2 \right) \left(\mathfrak{d}_1^1 \mathfrak{w}_3^1 + \mathfrak{d}_1^2 \mathfrak{w}_3^2 \right) \left(\mathfrak{d}_3^1 \mathfrak{w}_1^1 + \mathfrak{d}_3^2 \mathfrak{w}_1^2 \right) + \\
& - \frac{1}{\alpha^2} \left(\mathfrak{d}_3^1 \mathfrak{w}_3^1 + \mathfrak{d}_3^2 \mathfrak{w}_3^2 \right) \left(\mathfrak{d}_1^1 \mathfrak{w}_2^1 + \mathfrak{d}_1^2 \mathfrak{w}_2^2 \right) \left(\mathfrak{d}_2^1 \mathfrak{w}_1^1 + \mathfrak{d}_2^2 \mathfrak{w}_1^2 \right)
\end{aligned}$$

The previous expression can be written as:

$$(\mathfrak{f}_3 - 1) \alpha = \mathfrak{b}_0 + \frac{1}{\alpha} \mathfrak{b}_1 + \frac{1}{\alpha^2} \mathfrak{b}_2$$

3.1.3.1 Coefficient 0

$$\begin{aligned}
b_0 &= \\
&+ \left(d_1^1 w_1^1 + d_1^2 w_1^2 \right) + \\
&+ \left(d_2^1 w_2^1 + d_2^2 w_2^2 \right) + \\
&+ \left(d_3^1 w_3^1 + d_3^2 w_3^2 \right)
\end{aligned}$$

$$b_0 = \left(d^1 \right)^T w^1 + \left(d^2 \right)^T w^2$$

3.1.3.2 Coefficient 1

$$\begin{aligned}
b_1 &= \\
&+ \left(d_1^1 w_1^1 + d_1^2 w_1^2 \right) \left(d_2^1 w_2^1 + d_2^2 w_2^2 \right) + \\
&- \left(d_1^1 w_2^1 + d_1^2 w_2^2 \right) \left(d_2^1 w_1^1 + d_2^2 w_1^2 \right) + \\
&+ \left(d_1^1 w_1^1 + d_1^2 w_1^2 \right) \left(d_3^1 w_3^1 + d_3^2 w_3^2 \right) + \\
&- \left(d_1^1 w_3^1 + d_1^2 w_3^2 \right) \left(d_3^1 w_1^1 + d_3^2 w_1^2 \right) + \\
&+ \left(d_2^1 w_2^1 + d_2^2 w_2^2 \right) \left(d_3^1 w_3^1 + d_3^2 w_3^2 \right) + \\
&- \left(d_2^1 w_3^1 + d_2^2 w_3^2 \right) \left(d_3^1 w_2^1 + d_3^2 w_2^2 \right)
\end{aligned}$$

$$b_1 = \left(d^1 \times d^2 \right)^T \left(w^1 \times w^2 \right)$$

3.1.3.3 Coefficient 2

$$\begin{aligned}
b_2 &= \\
&+ \left(d_1^1 w_1^1 + d_1^2 w_1^2 \right) \left(d_2^1 w_2^1 + d_2^2 w_2^2 \right) \left(d_3^1 w_3^1 + d_3^2 w_3^2 \right) + \\
&+ \left(d_1^1 w_2^1 + d_1^2 w_2^2 \right) \left(d_2^1 w_3^1 + d_2^2 w_3^2 \right) \left(d_3^1 w_1^1 + d_3^2 w_1^2 \right) + \\
&+ \left(d_1^1 w_3^1 + d_1^2 w_3^2 \right) \left(d_2^1 w_1^1 + d_2^2 w_1^2 \right) \left(d_3^1 w_2^1 + d_3^2 w_2^2 \right) + \\
&- \left(d_1^1 w_1^1 + d_1^2 w_1^2 \right) \left(d_2^1 w_3^1 + d_2^2 w_3^2 \right) \left(d_3^1 w_2^1 + d_3^2 w_2^2 \right) + \\
&- \left(d_2^1 w_2^1 + d_2^2 w_2^2 \right) \left(d_1^1 w_3^1 + d_1^2 w_3^2 \right) \left(d_3^1 w_1^1 + d_3^2 w_1^2 \right) + \\
&- \left(d_3^1 w_3^1 + d_3^2 w_3^2 \right) \left(d_1^1 w_2^1 + d_1^2 w_2^2 \right) \left(d_2^1 w_1^1 + d_2^2 w_1^2 \right)
\end{aligned}$$

$$b_2 = 0$$

3.1.3.4 Final expression

$$(\mathbf{f}_3 - 1) \alpha = (\mathbf{d}^1)^T \mathbf{w}^1 + (\mathbf{d}^2)^T \mathbf{w}^2 + \frac{1}{\alpha} (\mathbf{d}^1 \times \mathbf{d}^2)^T (\mathbf{w}^1 \times \mathbf{w}^2)$$

$$\mathbf{w}^i = \mathbf{w} \times \mathbf{v}^i \Rightarrow$$

$$(\mathbf{f}_3 - 1) \alpha = (\mathbf{d}^1)^T (\mathbf{w} \times \mathbf{v}^1) + (\mathbf{d}^2)^T (\mathbf{w} \times \mathbf{v}^2) + \frac{1}{\alpha} (\mathbf{d}^1 \times \mathbf{d}^2)^T \mathbf{w}^T (\mathbf{v}^1 \times \mathbf{v}^2) \mathbf{w}$$

$$\left. \begin{aligned} \mathbf{d}^1 &= \bar{\mathbf{v}}^2 - \mathbf{v}^2 \\ \mathbf{d}^2 &= \mathbf{v}^1 - \bar{\mathbf{v}}^1 \end{aligned} \right\} \Rightarrow$$

$$\begin{aligned} (\mathbf{f}_3 - 1) \alpha &= \frac{\mathbf{w}^T (\mathbf{v}^1 \times \mathbf{v}^2)}{\alpha} \mathbf{w}^T (\bar{\mathbf{v}}^1 \times \bar{\mathbf{v}}^2) - \mathbf{w}^T (\mathbf{v}^1 \times \mathbf{v}^2) + \\ &+ \left[1 - \frac{\mathbf{w}^T (\mathbf{v}^1 \times \mathbf{v}^2)}{\alpha} \right] \left[(\bar{\mathbf{v}}^2)^T (\mathbf{w} \times \mathbf{v}^1) - (\bar{\mathbf{v}}^1)^T (\mathbf{w} \times \mathbf{v}^2) - \mathbf{w}^T (\mathbf{v}^1 \times \mathbf{v}^2) \right] \end{aligned}$$

$$\alpha = \mathbf{w}^T (\mathbf{v}^1 \times \mathbf{v}^2) \Rightarrow$$

$$\mathbf{f}_3 = \frac{1}{\alpha} \mathbf{w}^T (\bar{\mathbf{v}}^1 \times \bar{\mathbf{v}}^2)$$

$$\bar{\alpha} \bar{\mathbf{w}} = \bar{\mathbf{v}}^1 \times \bar{\mathbf{v}}^2 \Rightarrow$$

$$\mathbf{f}_3 = \frac{\bar{\alpha}}{\alpha} \mathbf{w}^T \bar{\mathbf{w}}$$

$$\bar{\mathbf{w}} = \mathbf{w} \Rightarrow \mathbf{f}_3 = \frac{\bar{\alpha}}{\alpha}$$

Notice that, as expected, the invariant 3 can be interpreted as the deformed volume divided by the undeformed volume.

$$\mathbf{f}_3 = \frac{\bar{\alpha} \delta}{\alpha \delta}$$

3.2 Right Cauchy-Green deformation tensor

The right Cauchy-Green deformation tensor can be written in terms of the deformation gradient tensor as:

$$\mathbf{C} = \mathbf{F}^T \mathbf{F}$$

The invariants of the right Cauchy-Green deformation tensor can be written as:

3.2.1 Invariant 1

$$c_1 = \text{tr}(\mathbf{C}) = \text{tr}(\mathbf{F}^T \mathbf{F}) = f_2$$

$$c_1 = 1 + \frac{1}{\alpha^2} \left[(\bar{\mathbf{v}}^2)^T \bar{\mathbf{z}}^1 - (\bar{\mathbf{v}}^1)^T \bar{\mathbf{z}}^2 \right]$$

3.2.2 Invariant 2

$$c_2 = \text{tr}(\mathbf{C}^T \mathbf{C}) = \text{tr}(\mathbf{F}^T \mathbf{F} \mathbf{F}^T \mathbf{F})$$

$$\mathbf{F} = \mathbf{I} + \frac{1}{\alpha} \left[\mathbf{d}^1 (\mathbf{w}^1)^T + \mathbf{d}^2 (\mathbf{w}^2)^T \right]$$

$$\mathbf{A} = \left[\mathbf{d}^1 (\mathbf{w}^1)^T + \mathbf{d}^2 (\mathbf{w}^2)^T \right] \Rightarrow$$

$$\begin{aligned} & \left[\text{tr}(\mathbf{F}^T \mathbf{F} \mathbf{F}^T \mathbf{F}) - 3 \right] \alpha^4 = \\ & + 4\alpha^3 \text{tr}(\mathbf{A}) + 2\alpha^2 \text{tr}(\mathbf{A} \mathbf{A}) + 4\alpha^2 \text{tr}(\mathbf{A}^T \mathbf{A}) + 4\alpha \text{tr}(\mathbf{A} \mathbf{A}^T \mathbf{A}) + \text{tr}(\mathbf{A}^T \mathbf{A} \mathbf{A}^T \mathbf{A}) \end{aligned}$$

The expression for the Invariant 2 can be written as:

$$(c_2 - 3) \alpha^4 = 4\alpha^3 b_0 + 2\alpha^2 b_1 + 4\alpha^2 b_2 + 4\alpha b_3 + b_4$$

3.2.2.1 Coefficient 0

$$b_0 = \text{tr}(\mathbf{A})$$

$$b_0 = (\mathbf{w}^1)^T \bar{\mathbf{v}}^2 - (\mathbf{w}^2)^T \bar{\mathbf{v}}^1 - 2\alpha$$

3.2.2.2 Coefficient 1

$$b_1 = \text{tr}(\mathbf{A}\mathbf{A})$$

$$\begin{aligned} b_1 = & \\ & +2\alpha \left(\mathbf{w}^2\right)^{\text{T}} \bar{\mathbf{v}}^1 - 2\alpha \left(\mathbf{w}^1\right)^{\text{T}} \bar{\mathbf{v}}^2 + 2\alpha^2 + \\ & + \left(\mathbf{w}^2\right)^{\text{T}} \bar{\mathbf{v}}^1 \left(\mathbf{w}^2\right)^{\text{T}} \bar{\mathbf{v}}^1 - 2 \left(\mathbf{w}^1\right)^{\text{T}} \bar{\mathbf{v}}^1 \left(\mathbf{w}^2\right)^{\text{T}} \bar{\mathbf{v}}^2 + \left(\mathbf{w}^1\right)^{\text{T}} \bar{\mathbf{v}}^2 \left(\mathbf{w}^1\right)^{\text{T}} \bar{\mathbf{v}}^2 \end{aligned}$$

3.2.2.3 Coefficient 2

$$b_2 = \text{tr}(\mathbf{A}^{\text{T}}\mathbf{A})$$

$$\begin{aligned} b_2 = & \\ & +2\alpha \left(\mathbf{w}^2\right)^{\text{T}} \bar{\mathbf{v}}^1 - 2\alpha \left(\mathbf{w}^1\right)^{\text{T}} \bar{\mathbf{v}}^2 + 2\alpha^2 + \\ & + \left(\mathbf{v}^2\right)^{\text{T}} \mathbf{v}^2 \left(\bar{\mathbf{v}}^1\right)^{\text{T}} \bar{\mathbf{v}}^1 - 2 \left(\bar{\mathbf{v}}^1\right)^{\text{T}} \bar{\mathbf{v}}^2 \left(\mathbf{v}^1\right)^{\text{T}} \mathbf{v}^2 + \left(\mathbf{v}^1\right)^{\text{T}} \mathbf{v}^1 \left(\bar{\mathbf{v}}^2\right)^{\text{T}} \bar{\mathbf{v}}^2 \end{aligned}$$

3.2.2.4 Coefficient 3

$$b_3 = \text{tr}(\mathbf{A}\mathbf{A}^{\text{T}}\mathbf{A})$$

$$\begin{aligned}
b_3 = & -2\alpha^3 + \\
& +3\alpha^2 \left(\bar{\mathbf{v}}^2\right)^T \mathbf{w}^1 - 3\alpha^2 \left(\bar{\mathbf{v}}^1\right)^T \mathbf{w}^2 + \\
& -\alpha \left[\left(\mathbf{v}^2\right)^T \mathbf{v}^2 \left(\bar{\mathbf{v}}^1\right)^T \bar{\mathbf{v}}^1 - 2 \left(\mathbf{v}^1\right)^T \mathbf{v}^2 \left(\bar{\mathbf{v}}^2\right)^T \bar{\mathbf{v}}^1 + \left(\mathbf{v}^1\right)^T \mathbf{v}^1 \left(\bar{\mathbf{v}}^2\right)^T \bar{\mathbf{v}}^2 \right] + \\
& -\alpha \left[\left(\mathbf{w}^2\right)^T \bar{\mathbf{v}}^1 \right]^2 + 2\alpha \left(\mathbf{w}^1\right)^T \bar{\mathbf{v}}^1 \left(\mathbf{w}^2\right)^T \bar{\mathbf{v}}^2 - \alpha \left[\left(\mathbf{w}^1\right)^T \bar{\mathbf{v}}^2 \right]^2 + \\
& + \left(\bar{\mathbf{v}}^2\right)^T \mathbf{v}^2 \left(\mathbf{v}^2\right)^T \mathbf{v}^1 \left(\mathbf{w}^1\right)^T \bar{\mathbf{v}}^1 - \left(\bar{\mathbf{v}}^1\right)^T \mathbf{v}^2 \left(\mathbf{v}^2\right)^T \mathbf{v}^2 \left(\mathbf{w}^1\right)^T \bar{\mathbf{v}}^1 + \\
& + \left(\bar{\mathbf{v}}^1\right)^T \mathbf{v}^2 \left(\mathbf{v}^1\right)^T \mathbf{v}^2 \left(\mathbf{w}^1\right)^T \bar{\mathbf{v}}^2 - \left(\bar{\mathbf{v}}^2\right)^T \mathbf{v}^2 \left(\mathbf{v}^1\right)^T \mathbf{v}^1 \left(\mathbf{w}^1\right)^T \bar{\mathbf{v}}^2 + \\
& + \left(\bar{\mathbf{v}}^1\right)^T \mathbf{v}^1 \left(\mathbf{v}^2\right)^T \mathbf{v}^2 \left(\mathbf{w}^2\right)^T \bar{\mathbf{v}}^1 - \left(\bar{\mathbf{v}}^2\right)^T \mathbf{v}^1 \left(\mathbf{v}^2\right)^T \mathbf{v}^1 \left(\mathbf{w}^2\right)^T \bar{\mathbf{v}}^1 + \\
& + \left(\bar{\mathbf{v}}^2\right)^T \mathbf{v}^1 \left(\mathbf{v}^1\right)^T \mathbf{v}^1 \left(\mathbf{w}^2\right)^T \bar{\mathbf{v}}^2 - \left(\bar{\mathbf{v}}^1\right)^T \mathbf{v}^1 \left(\mathbf{v}^1\right)^T \mathbf{v}^2 \left(\mathbf{w}^2\right)^T \bar{\mathbf{v}}^2 + \\
& + \left(\bar{\mathbf{v}}^2\right)^T \bar{\mathbf{v}}^1 \left(\mathbf{v}^2\right)^T \mathbf{v}^2 \left(\mathbf{w}^1\right)^T \bar{\mathbf{v}}^1 - \left(\bar{\mathbf{v}}^2\right)^T \bar{\mathbf{v}}^2 \left(\mathbf{v}^2\right)^T \mathbf{v}^1 \left(\mathbf{w}^1\right)^T \bar{\mathbf{v}}^1 + \\
& + \left(\bar{\mathbf{v}}^2\right)^T \bar{\mathbf{v}}^2 \left(\mathbf{v}^1\right)^T \mathbf{v}^1 \left(\mathbf{w}^1\right)^T \bar{\mathbf{v}}^2 - \left(\bar{\mathbf{v}}^2\right)^T \bar{\mathbf{v}}^1 \left(\mathbf{v}^1\right)^T \mathbf{v}^2 \left(\mathbf{w}^1\right)^T \bar{\mathbf{v}}^2 + \\
& + \left(\bar{\mathbf{v}}^2\right)^T \bar{\mathbf{v}}^1 \left(\mathbf{v}^2\right)^T \mathbf{v}^1 \left(\mathbf{w}^2\right)^T \bar{\mathbf{v}}^1 - \left(\bar{\mathbf{v}}^1\right)^T \bar{\mathbf{v}}^1 \left(\mathbf{v}^2\right)^T \mathbf{v}^2 \left(\mathbf{w}^2\right)^T \bar{\mathbf{v}}^1 + \\
& + \left(\bar{\mathbf{v}}^1\right)^T \bar{\mathbf{v}}^1 \left(\mathbf{v}^1\right)^T \mathbf{v}^2 \left(\mathbf{w}^2\right)^T \bar{\mathbf{v}}^2 - \left(\bar{\mathbf{v}}^2\right)^T \bar{\mathbf{v}}^1 \left(\mathbf{v}^1\right)^T \mathbf{v}^1 \left(\mathbf{w}^2\right)^T \bar{\mathbf{v}}^2
\end{aligned}$$

3.2.2.5 Coefficient 4

$$b_4 = \text{tr} \left(\mathbf{A}^T \mathbf{A} \mathbf{A}^T \mathbf{A} \right)$$

[illegible]

3.2.2.6 Final expression

$$\begin{aligned}
& (c_2 - 1) \alpha^4 = \\
& + 2\alpha^2 (\mathbf{v}^1)^T \mathbf{v}^1 (\bar{\mathbf{v}}^2)^T \bar{\mathbf{v}}^2 + 2\alpha^2 (\mathbf{v}^2)^T \mathbf{v}^2 (\bar{\mathbf{v}}^1)^T \bar{\mathbf{v}}^1 + \\
& - 4\alpha^2 (\mathbf{v}^1)^T \mathbf{v}^2 (\bar{\mathbf{v}}^1)^T \bar{\mathbf{v}}^2 + \\
& + 4 (\mathbf{v}^1)^T \mathbf{v}^2 (\mathbf{v}^2)^T \mathbf{v}^2 (\bar{\mathbf{v}}^1)^T \mathbf{v}^1 (\bar{\mathbf{v}}^1)^T \mathbf{v}^2 + 4 (\mathbf{v}^1)^T \mathbf{v}^1 (\mathbf{v}^1)^T \mathbf{v}^2 (\bar{\mathbf{v}}^2)^T \mathbf{v}^1 (\bar{\mathbf{v}}^2)^T \mathbf{v}^2 + \\
& + 4 (\mathbf{v}^1)^T \mathbf{v}^1 (\mathbf{v}^1)^T \mathbf{v}^2 (\bar{\mathbf{v}}^1)^T \mathbf{v}^2 (\bar{\mathbf{v}}^2)^T \mathbf{v}^2 + 4 (\mathbf{v}^1)^T \mathbf{v}^2 (\mathbf{v}^2)^T \mathbf{v}^2 (\bar{\mathbf{v}}^1)^T \mathbf{v}^1 (\bar{\mathbf{v}}^2)^T \mathbf{v}^1 + \\
& - 2 (\mathbf{v}^1)^T \mathbf{v}^1 (\mathbf{v}^2)^T \mathbf{v}^2 (\bar{\mathbf{v}}^1)^T \mathbf{v}^2 (\bar{\mathbf{v}}^1)^T \mathbf{v}^2 - 2 (\mathbf{v}^1)^T \mathbf{v}^1 (\mathbf{v}^2)^T \mathbf{v}^2 (\bar{\mathbf{v}}^2)^T \mathbf{v}^1 (\bar{\mathbf{v}}^2)^T \mathbf{v}^1 + \\
& - 2 (\mathbf{v}^1)^T \mathbf{v}^1 (\mathbf{v}^1)^T \mathbf{v}^1 (\bar{\mathbf{v}}^2)^T \mathbf{v}^2 (\bar{\mathbf{v}}^2)^T \mathbf{v}^2 - 2 (\mathbf{v}^2)^T \mathbf{v}^2 (\mathbf{v}^2)^T \mathbf{v}^2 (\bar{\mathbf{v}}^1)^T \mathbf{v}^1 (\bar{\mathbf{v}}^1)^T \mathbf{v}^1 + \\
& - 4 (\mathbf{v}^1)^T \mathbf{v}^2 (\mathbf{v}^1)^T \mathbf{v}^2 (\bar{\mathbf{v}}^1)^T \mathbf{v}^2 (\bar{\mathbf{v}}^2)^T \mathbf{v}^1 + \\
& - 4 (\mathbf{v}^1)^T \mathbf{v}^2 (\mathbf{v}^1)^T \mathbf{v}^2 (\bar{\mathbf{v}}^1)^T \mathbf{v}^1 (\bar{\mathbf{v}}^2)^T \mathbf{v}^2 + \\
& + (\mathbf{v}^1)^T \mathbf{v}^1 (\mathbf{v}^1)^T \mathbf{v}^1 (\bar{\mathbf{v}}^2)^T \bar{\mathbf{v}}^2 (\bar{\mathbf{v}}^2)^T \bar{\mathbf{v}}^2 + (\mathbf{v}^2)^T \mathbf{v}^2 (\mathbf{v}^2)^T \mathbf{v}^2 (\bar{\mathbf{v}}^1)^T \bar{\mathbf{v}}^1 (\bar{\mathbf{v}}^1)^T \bar{\mathbf{v}}^1 + \\
& - 4 (\mathbf{v}^1)^T \mathbf{v}^1 (\mathbf{v}^1)^T \mathbf{v}^2 (\bar{\mathbf{v}}^1)^T \bar{\mathbf{v}}^2 (\bar{\mathbf{v}}^2)^T \bar{\mathbf{v}}^2 - 4 (\mathbf{v}^1)^T \mathbf{v}^2 (\mathbf{v}^2)^T \mathbf{v}^2 (\bar{\mathbf{v}}^1)^T \bar{\mathbf{v}}^1 (\bar{\mathbf{v}}^1)^T \bar{\mathbf{v}}^2 + \\
& + 2 (\mathbf{v}^1)^T \mathbf{v}^2 (\mathbf{v}^1)^T \mathbf{v}^2 (\bar{\mathbf{v}}^1)^T \bar{\mathbf{v}}^1 (\bar{\mathbf{v}}^2)^T \bar{\mathbf{v}}^2 + \\
& + 2 (\mathbf{v}^1)^T \mathbf{v}^1 (\mathbf{v}^2)^T \mathbf{v}^2 (\bar{\mathbf{v}}^1)^T \bar{\mathbf{v}}^2 (\bar{\mathbf{v}}^1)^T \bar{\mathbf{v}}^2 + \\
& + 2 (\mathbf{v}^1)^T \mathbf{v}^2 (\mathbf{v}^1)^T \mathbf{v}^2 (\bar{\mathbf{v}}^1)^T \bar{\mathbf{v}}^2 (\bar{\mathbf{v}}^1)^T \bar{\mathbf{v}}^2
\end{aligned}$$

The previous expression can be written as:

$$\begin{aligned}
& c_2 = 1 + \\
& + \frac{2}{\alpha^2} (\mathbf{w}^T \bar{\mathbf{z}}^1 \mathbf{w}^T \bar{\mathbf{v}}^2 - \mathbf{w}^T \bar{\mathbf{z}}^2 \mathbf{w}^T \bar{\mathbf{v}}^1) + \\
& + \frac{1}{\alpha^4} \left[(\bar{\mathbf{v}}^1)^T \bar{\mathbf{v}}^1 (\bar{\mathbf{z}}^2)^T \bar{\mathbf{z}}^2 - (\bar{\mathbf{v}}^1)^T \bar{\mathbf{v}}^2 (\bar{\mathbf{z}}^2)^T \bar{\mathbf{z}}^1 \right] + \\
& + \frac{1}{\alpha^4} \left[(\bar{\mathbf{v}}^2)^T \bar{\mathbf{v}}^2 (\bar{\mathbf{z}}^1)^T \bar{\mathbf{z}}^1 - (\bar{\mathbf{v}}^1)^T \bar{\mathbf{v}}^2 (\bar{\mathbf{z}}^2)^T \bar{\mathbf{z}}^1 \right]
\end{aligned}$$

3.2.3 Invariant 3

$$c_3 = \det(\mathbf{C}) = \det(\mathbf{F}^T \mathbf{F}) = \det^2(\mathbf{F}) = \mathbf{f}_3^2$$

$$c_3 = \left(\frac{\bar{\alpha}}{\alpha} \right)^2$$

4 Plane stress

4.1 Deformation gradient tensor

$$\hat{\mathbf{F}} = \mathbf{F} + \left(\frac{\bar{\delta}}{\delta} \bar{\mathbf{w}} - \mathbf{w} \right) \mathbf{w}^T$$

Notice that,

$$\hat{\mathbf{F}} \mathbf{v}^1 = \mathbf{F} \mathbf{v}^1 + \left(\frac{\bar{\delta}}{\delta} \bar{\mathbf{w}} - \mathbf{w} \right) \mathbf{w}^T \mathbf{v}^1 = \bar{\mathbf{v}}^1$$

$$\hat{\mathbf{F}} \mathbf{v}^2 = \mathbf{F} \mathbf{v}^2 + \left(\frac{\bar{\delta}}{\delta} \bar{\mathbf{w}} - \mathbf{w} \right) \mathbf{w}^T \mathbf{v}^2 = \bar{\mathbf{v}}^2$$

$$\hat{\mathbf{F}} \mathbf{v}^3 = \mathbf{F} \mathbf{v}^3 + \left(\frac{\bar{\delta}}{\delta} \bar{\mathbf{w}} - \mathbf{w} \right) \mathbf{w}^T \mathbf{v}^3 = \bar{\mathbf{v}}^3$$

$$\hat{\mathbf{F}} (\delta \mathbf{w}) = (\bar{\delta} \bar{\mathbf{w}})$$

4.1.1 Invariant 1

$$\hat{f}_1 = \text{tr} (\hat{\mathbf{F}})$$

$$\hat{\mathbf{F}} = \mathbf{F} + \left(\frac{\bar{\delta}}{\delta} \bar{\mathbf{w}} - \mathbf{w} \right) \mathbf{w}^T$$

$$\hat{f}_1 = f_1 + \frac{\bar{\delta}}{\delta} \mathbf{w}^T \bar{\mathbf{w}} - 1$$

$$\hat{f}_1 = \frac{1}{\alpha} \left[\left(\mathbf{w}^1 \right)^T \bar{\mathbf{v}}^2 - \left(\mathbf{w}^2 \right)^T \bar{\mathbf{v}}^1 \right] + \frac{\bar{\delta}}{\delta} \mathbf{w}^T \bar{\mathbf{w}}$$

4.1.2 Invariant 2

$$\hat{f}_2 = \text{tr} \left(\hat{F}^T \hat{F} \right)$$

$$\hat{F} = F + \left(\frac{\bar{\delta}}{\delta} \bar{w} - w \right) w^T$$

$$A = \left(\frac{\bar{\delta}}{\delta} \bar{w} - w \right) w^T \Rightarrow$$

$$\hat{f}_2 = f_2 + 2\text{tr} \left(F^T A \right) + \text{tr} \left(A^T A \right)$$

$$\text{tr} \left(F^T A \right) = \frac{\bar{\delta}}{\delta} w^T \bar{w} - 1$$

$$\text{tr} \left(A^T A \right) = \left(\frac{\bar{\delta}}{\delta} \right)^2 - 2 \frac{\bar{\delta}}{\delta} w^T \bar{w} + 1$$

$$\hat{f}_2 = f_2 + \left(\frac{\bar{\delta}}{\delta} \right)^2 - 1$$

$$\hat{f}_2 = \frac{1}{\alpha^2} \left[\left(\bar{v}^2 \right)^T \bar{z}^1 - \left(\bar{v}^1 \right)^T \bar{z}^2 \right] + \left(\frac{\bar{\delta}}{\delta} \right)^2$$

4.1.3 Invariant 3

$$\hat{f}_3 = \det \left(\hat{F} \right) = \hat{f}_{11} \hat{f}_{22} \hat{f}_{33} + \hat{f}_{12} \hat{f}_{23} \hat{f}_{31} + \hat{f}_{13} \hat{f}_{32} \hat{f}_{21} - \hat{f}_{31} \hat{f}_{22} \hat{f}_{13} - \hat{f}_{32} \hat{f}_{23} \hat{f}_{11} - \hat{f}_{33} \hat{f}_{12} \hat{f}_{21}$$

$$\hat{f}_{ij} = f_{ij} + \frac{\bar{\delta}}{\delta} \bar{w}_i w_j - w_i w_j$$

$$\begin{aligned}
\hat{\mathbf{f}}_3 &= \frac{1}{\alpha} \mathbf{w}^T (\bar{\mathbf{v}}^1 \times \bar{\mathbf{v}}^2) + \\
&- \frac{1}{\alpha^2} \left[\mathbf{w}^T (\mathbf{d}^1 \times \mathbf{d}^2) \mathbf{w}^T (\mathbf{w}^1 \times \mathbf{w}^2) + \alpha^2 \mathbf{w}^T \mathbf{w} \right] + \\
&+ \frac{1}{\alpha} \left[\mathbf{w}^T \mathbf{d}^1 \mathbf{w}^T \mathbf{w}^1 + \mathbf{w}^T \mathbf{d}^2 \mathbf{w}^T \mathbf{w}^2 - (\mathbf{w}^1)^T \mathbf{d}^1 - (\mathbf{w}^2)^T \mathbf{d}^2 \right] + \\
&+ \frac{1}{\alpha^2} \left\{ \bar{\mathbf{w}}^T (\mathbf{d}^1 \times \mathbf{d}^2) \mathbf{w}^T (\mathbf{w}^1 \times \mathbf{w}^2) + \alpha^2 \bar{\mathbf{w}}^T \mathbf{w} + \right. \\
&\left. + \alpha (\mathbf{d}^1)^T [\bar{\mathbf{w}} \times (\mathbf{w}^1 \times \mathbf{w})] + \alpha (\mathbf{d}^2)^T [\bar{\mathbf{w}} \times (\mathbf{w}^2 \times \mathbf{w})] \right\} \frac{\bar{\delta}}{\delta}
\end{aligned}$$

$$\mathbf{w}^i = \mathbf{w} \times \mathbf{v}^i \Rightarrow$$

$$\begin{aligned}
\hat{\mathbf{f}}_3 &= \frac{1}{\alpha} \mathbf{w}^T (\bar{\mathbf{v}}^1 \times \bar{\mathbf{v}}^2) + \\
&- \frac{1}{\alpha^2} \mathbf{w}^T (\mathbf{d}^1 \times \mathbf{d}^2) \mathbf{w}^T (\mathbf{v}^1 \times \mathbf{v}^2) - 1 + \\
&- \frac{1}{\alpha} \mathbf{w}^T (\mathbf{v}^1 \times \mathbf{d}^1) - \frac{1}{\alpha} \mathbf{w}^T (\mathbf{v}^2 \times \mathbf{d}^2) + \\
&+ \left\{ \frac{1}{\alpha^2} \bar{\mathbf{w}}^T (\mathbf{d}^1 \times \mathbf{d}^2) \mathbf{w}^T (\mathbf{v}^1 \times \mathbf{v}^2) + \bar{\mathbf{w}}^T \mathbf{w} + \right. \\
&\left. + \frac{1}{\alpha} \bar{\mathbf{w}}^T (\mathbf{v}^1 \times \mathbf{d}^1) + \frac{1}{\alpha} \bar{\mathbf{w}}^T (\mathbf{v}^2 \times \mathbf{d}^2) \right\} \frac{\bar{\delta}}{\delta}
\end{aligned}$$

$$\alpha \mathbf{w} = \mathbf{v}^1 \times \mathbf{v}^2 \Rightarrow$$

$$\begin{aligned}
\hat{\mathbf{f}}_3 &= \frac{1}{\alpha} \mathbf{w}^T (\bar{\mathbf{v}}^1 \times \bar{\mathbf{v}}^2) - 1 + \\
&- \frac{1}{\alpha} \mathbf{w}^T \left[(\mathbf{d}^1 \times \mathbf{d}^2) + (\mathbf{v}^1 \times \mathbf{d}^1) + (\mathbf{v}^2 \times \mathbf{d}^2) \right] + \\
&+ \frac{1}{\alpha} \frac{\bar{\delta}}{\delta} \bar{\mathbf{w}}^T \left[(\mathbf{d}^1 \times \mathbf{d}^2) + (\mathbf{v}^1 \times \mathbf{d}^1) + (\mathbf{v}^2 \times \mathbf{d}^2) + (\mathbf{v}^1 \times \mathbf{v}^2) \right]
\end{aligned}$$

$$(\bar{\mathbf{v}}^1 \times \bar{\mathbf{v}}^2) = (\mathbf{v}^1 \times \mathbf{v}^2) + (\mathbf{v}^1 \times \mathbf{d}^1) + (\mathbf{v}^2 \times \mathbf{d}^2) + (\mathbf{d}^1 \times \mathbf{d}^2) \Rightarrow$$

$$\hat{\mathbf{f}}_3 = \frac{\bar{\delta}}{\delta} \frac{\bar{\mathbf{w}}^T (\bar{\mathbf{v}}^1 \times \bar{\mathbf{v}}^2)}{\alpha}$$

$$\bar{\alpha} \bar{\mathbf{w}} = \bar{\mathbf{v}}^1 \times \bar{\mathbf{v}}^2 \Rightarrow$$

$$\hat{f}_3 = \frac{\bar{\delta}\bar{\alpha}}{\delta\alpha}$$

Notice that, as expected, the invariant 3 can be interpreted as the deformed volume divided by the undeformed volume.

4.2 Right Cauchy-Green deformation tensor

The right Cauchy-Green deformation tensor can be written in terms of the deformation gradient tensor as:

$$\hat{C} = \hat{F}^T \hat{F}$$

The invariants of the right Cauchy-Green deformation tensor can be written as:

4.2.1 Invariant 1

$$\hat{c}_1 = \text{tr}(\hat{C}) = \text{tr}(\hat{F}^T \hat{F}) = \hat{f}_2$$

$$\hat{c}_1 = f_2 + \left(\frac{\bar{\delta}}{\delta}\right)^2 - 1$$

$$\hat{c}_1 = \frac{1}{\alpha^2} \left[(\bar{v}^2)^T \bar{z}^1 - (\bar{v}^1)^T \bar{z}^2 \right] + \left(\frac{\bar{\delta}}{\delta}\right)^2$$

4.2.2 Invariant 2

$$\hat{c}_2 = \text{tr}(\hat{C}^T \hat{C}) = \text{tr}(\hat{F}^T \hat{F} \hat{F}^T \hat{F})$$

$$\hat{F} = F + \left(\frac{\bar{\delta}}{\delta} \bar{w} - w\right) w^T$$

$$A = \left(\frac{\bar{\delta}}{\delta} \bar{w} - w\right) w^T \Rightarrow$$

$$\begin{aligned}
\hat{c}_2 &= c_2 + \\
&+ \text{tr} \left(F^T F F^T A \right) + \text{tr} \left(F^T F A^T F \right) + \text{tr} \left(F^T F A^T A \right) + \\
&+ \text{tr} \left(F^T A F^T F \right) + \text{tr} \left(F^T A F^T A \right) + \text{tr} \left(F^T A A^T F \right) + \text{tr} \left(F^T A A^T A \right) + \\
&+ \text{tr} \left(A^T F F^T F \right) + \text{tr} \left(A^T F F^T A \right) + \text{tr} \left(A^T F A^T F \right) + \text{tr} \left(A^T F A^T A \right) + \\
&+ \text{tr} \left(A^T A F^T F \right) + \text{tr} \left(A^T A F^T A \right) + \text{tr} \left(A^T A A^T F \right) + \text{tr} \left(A^T A A^T A \right)
\end{aligned}$$

$$\begin{aligned}
\hat{c}_2 &= c_2 + \\
&+ \text{tr} \left(A^T A A^T A \right) + 4 \text{tr} \left(F^T A F^T F \right) + 4 \text{tr} \left(A^T A F^T A \right) + \\
&+ 2 \text{tr} \left[\left(F^T A \right) \left(F^T A \right)^T \right] + 2 \text{tr} \left(F^T A F^T A \right) + 2 \text{tr} \left(A^T A F^T F \right)
\end{aligned}$$

$$\text{tr} \left(A^T A A^T A \right) = \left[\left(\frac{\bar{\delta}}{\delta} \right)^2 - 2 \frac{\bar{\delta}}{\delta} w^T \bar{w} + 1 \right]^2$$

$$\text{tr} \left(F^T A F^T F \right) = \frac{\bar{\delta}}{\delta} \left(F^T w \right)^T \left(F^T \bar{w} \right) - \left(F^T w \right)^T \left(F^T w \right)$$

$$\text{tr} \left(A^T A F^T A \right) = \left[\left(\frac{\bar{\delta}}{\delta} \right)^2 - 2 \frac{\bar{\delta}}{\delta} w^T \bar{w} + 1 \right] \left(\frac{\bar{\delta}}{\delta} w^T \bar{w} - 1 \right)$$

$$\text{tr} \left[\left(F^T A \right) \left(F^T A \right)^T \right] = \left(\frac{\bar{\delta}}{\delta} \right)^2 \left(F^T \bar{w} \right)^T \left(F^T \bar{w} \right) - 2 \frac{\bar{\delta}}{\delta} \left(F^T \bar{w} \right)^T \left(F^T w \right) + \left(F^T w \right)^T \left(F^T w \right)$$

$$\text{tr} \left(F^T A F^T A \right) = \left(\frac{\bar{\delta}}{\delta} w^T \bar{w} - 1 \right)^2$$

$$\text{tr} \left(A^T A F^T F \right) = \left(\frac{\bar{\delta}}{\delta} \right)^2 - 2 \frac{\bar{\delta}}{\delta} w^T \bar{w} + 1$$

$$\left(F^T w \right)^T \left(F^T w \right) = 1 + \frac{1}{\alpha^2} \left(w^T \bar{v}^2 w^T \bar{z}^1 - w^T \bar{v}^1 w^T \bar{z}^2 \right)$$

$$\left(F^T \bar{w} \right)^T \left(F^T \bar{w} \right) = \left(w^T \bar{w} \right)^2$$

$$\hat{c}_2 = c_2 + \frac{2}{\alpha^2} \left(w^T \bar{v}^1 w^T \bar{z}^2 - w^T \bar{v}^2 w^T \bar{z}^1 \right) - 1 + \left(\frac{\bar{\delta}}{\delta} \right)^4$$

$$\begin{aligned}
\hat{c}_2 = & \\
& + \frac{1}{\alpha^4} \left[\left(\bar{v}^1 \right)^T \bar{v}^1 \left(\bar{z}^2 \right)^T \bar{z}^2 - \left(\bar{v}^1 \right)^T \bar{v}^2 \left(\bar{z}^2 \right)^T \bar{z}^1 \right] + \\
& + \frac{1}{\alpha^4} \left[\left(\bar{v}^2 \right)^T \bar{v}^2 \left(\bar{z}^1 \right)^T \bar{z}^1 - \left(\bar{v}^1 \right)^T \bar{v}^2 \left(\bar{z}^2 \right)^T \bar{z}^1 \right] + \\
& + \left(\frac{\bar{\delta}}{\delta} \right)^4
\end{aligned}$$

4.2.3 Invariant 3

$$\hat{c}_3 = \det (\hat{C}) = \det (\hat{F}^T \hat{F}) = \det^2 (\hat{F}) = \hat{f}_3^2$$

$$\hat{c}_3 = \left(\frac{\bar{\delta} \bar{\alpha}}{\delta \alpha} \right)^2$$

5 Appendix

$$a^T (b \times c) = b^T (c \times a) = c^T (a \times b)$$

$$a \times (b \times c) = (a^T c) b - (a^T b) c$$

$$\text{tr} (AB) = \text{tr} (BA)$$

$$\phi = a^T x \Rightarrow \frac{\partial \phi}{\partial x_i} = a_i$$

$$\phi = x^T x \Rightarrow \frac{\partial \phi}{\partial x_i} = 2x_i$$

$$\phi = a^T (b \times x) = (a \times b)^T x \Rightarrow \frac{\partial \phi}{\partial x_i} = (a \times b)_i$$

$$\det (AB) = \det (A) \det (B)$$

$$\det (A^T) = \det (A)$$

$$\det (A - \lambda I) = -\lambda^3 + \operatorname{tr} (A) \lambda^2 - \frac{1}{2} \left[\operatorname{tr}^2 (A) - \operatorname{tr} (AA) \right] \lambda + \det (A)$$